

SESHADRI CONSTANTS VIA TORIC DEGENERATIONS

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ABSTRACT. We give lower and upper bounds of Seshadri constants on toric varieties at any points. By using the lower bounds and toric degenerations, we can obtain some new computations or estimations of Seshadri constants on non-toric varieties. In particular, we investigate Seshadri constants on hypersurfaces in projective spaces and Fano 3-folds with Picard number one in detail.

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1. INTRODUCTION

In this paper, we study how to estimate Seshadri constants. First, we give lower and upper bounds of Seshadri constants on toric varieties at any points. Next, we obtain some new estimations of Seshadri constants on non-toric varieties by using toric degenerations. We consider varieties or schemes over the complex number field \mathbb{C} throughout this paper.

Demailly [Dem] defined an interesting invariant, Seshadri constant, which measures the local positivity of a line bundle on a projective variety:

Definition 1.1. Let L be a nef line bundle on a projective variety X , and take a (possibly singular) closed point $p \in X$. We define the Seshadri constant of L at p to be

$$\varepsilon(X, L; p) = \varepsilon(L; p) := \inf_C \left\{ \frac{C.L}{\text{mult}_p(C)} \right\},$$

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where C moves all reduced and irreducible curves on X passing through p , and $\text{mult}_p(C)$ is the multiplicity of C at p .

Remark 1.2. It is easily shown that $\varepsilon(L; p) = \max\{t \geq 0 \mid \mu^*L - tE \text{ is nef}\}$, where $\mu : \tilde{X} \rightarrow X$ is the blowing up at p and $E = \mu^{-1}(p)$ is the exceptional divisor (cf. [La2, Chapter 5]). Hence there is an inequality $\varepsilon(L; p) \leq \sqrt[n]{L^n / \text{mult}_p(X)}$ for any point $p \in X$, where n is the dimension of X .

For a subvariety Y of X , $\varepsilon(X, L; p) \leq \varepsilon(Y, L|_Y; p)$ holds for any $p \in Y \subset X$ by the definition of Seshadri constants. We will use this later repeatedly.

Seshadri constants sometimes have interesting geometric consequences. For example, lower bounds of Seshadri constants induce jet separations of adjoint linear series [Dem] and lower bounds of Gromov width (an invariant in symplectic geometry) [MP]. Upper bounds sometimes give fibrations or foliations [Na1], [Na2], [HW]. Seshadri constants are used to define the Ross-Thomas' slope stabilities for polarized varieties [RT].

But unfortunately it is very difficult to compute or estimate Seshadri constants in general. Many authors study about surfaces, but estimations in higher dimensional cases are very few. In higher dimensional cases, the following results are known:

In [EKL], Ein, Küchle, and Lazarsfeld show that $\varepsilon(X, L; p) \geq 1/\dim X$ holds for a very general point $p \in X$ for any polarized variety (X, L) . By [La1] and [Bau], lower bounds of Seshadri constants are obtained for abelian varieties. In [Di] or [BDH+], Seshadri constants on toric varieties at torus invariant points are computed. Somewhat surprisingly, we do not know how to compute the Seshadri constant on a polarized toric variety at a not necessarily torus invariant point in general.

In this paper, we investigate toric cases at first. Let M be a free abelian group of rank n and set $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$. For an integral polytope $P \subset M_{\mathbb{R}}$ of dimension n and a face σ of P , we will define positive real numbers $s_1(P; \sigma)$, $s_2(P; \sigma)$ and show:

Theorem 1.3 (=Theorem 3.17). *Let P be an integral polytope of dimension n in $M_{\mathbb{R}}$, and σ a face of P . Then,*

$$s_1(P; \sigma) \leq \varepsilon(X_P, L_P; p) \leq s_2(P; \sigma)$$

holds for any $p \in O_{\sigma}$, where (X_P, L_P) is the (normal) polarized toric variety defined by P , and $O_{\sigma} \subset X_P$ is the orbit corresponding to σ .

An important point is that $s_1(P; \sigma)$ and $s_2(P; \sigma)$ are computed or estimated more easily than $\varepsilon(X_P, L_P; p)$. Besides, $s_1(P; \sigma) = s_2(P; \sigma)$ often holds, thus we can compute $\varepsilon(X_P, L_P; p)$ explicitly in those cases.

Next, we study non-toric cases. In these cases, we mainly consider Seshadri constants at very general points (cf. [EKL]):

Definition 1.4. Let L be a nef line bundle on a projective variety X . The Seshadri constant $\varepsilon(X, L; 1)$ of L at a very general point is defined to be

$$\varepsilon(X, L; 1) := \varepsilon(X, L; p)$$

for a very general point $p \in X$.

Remark 1.5. In flat families, ampleness is an open condition in the base. Thus the map $p \mapsto \varepsilon(X, L; p)$ from the set of smooth closed points in X to \mathbb{R} has some lower-semicontinuity.

Hence $\varepsilon(X, L; p)$ does not depend on the choice of p if p is very general. (cf. [La2, Example 5.1.11])

The definition of Seshadri constants can be generalized to multi-points cases easily (cf. [La2, Definition 5.4.1], [BDH+, Definition 1.9]):

Definition 1.6. Let L be a nef line bundle on a projective variety X . For a positive integer r , $\overline{m} = (m_1, \dots, m_r) \in \mathbb{R}_{>0}^r$, and r points $p_1, \dots, p_r \in X$, the Seshadri constant $\varepsilon(X, L; m_1 p_1, \dots, m_r p_r)$ of L at p_1, \dots, p_r with wight \overline{m} is

$$\varepsilon(X, L; m_1 p_1, \dots, m_r p_r) := \inf_C \left\{ \frac{C \cdot L}{\sum_{i=1}^r m_i \text{mult}_{p_i}(C)} \right\},$$

where C moves all reduced and irreducible curves on X passing through at least one of p_1, \dots, p_r . In the same way as Remark 1.2, it holds that

$$\varepsilon(X, L; m_1 p_1, \dots, m_r p_r) = \max\{t \geq 0 \mid \mu^* L - t \sum_{i=1}^r m_i E_i \text{ is nef}\},$$

where $\mu : \tilde{X} \rightarrow X$ is the blowing up at p_1, \dots, p_r and $E_i = \mu^{-1}(p_i)$ is the exceptional divisor over p_i .

As Definition 1.4, we define the Seshadri constant $\varepsilon(X, L; \overline{m})$ of L at very general points with weight \overline{m} as follows:

$$\varepsilon(X, L; \overline{m}) = \varepsilon(X, L; m_1, \dots, m_r) := \varepsilon(X, L; m_1 p_1, \dots, m_r p_r)$$

for very general points $p_1, \dots, p_r \in X$.

Since Seshadri constants have some lower semicontinuties, degenerations are useful to get lower bounds of Seshadri constants. From Theorem 1.3, we obtain the following theorem:

Theorem 1.7 (special case of Corollary 4.4). *Let $f : \mathcal{X} \rightarrow T$ be a flat projective morphism over a smooth variety T with reduced and irreducible general fibers. Let \mathcal{L} be an f -ample line bundle on \mathcal{X} and $0 \in T$. Set $X_t = f^{-1}(t)$, $L_t = \mathcal{L}|_{X_t}$ for $t \in T$. If the normalization of the central fiber (X_0, L_0) is isomorphic to the polarized toric variety (X_P, L_P) for an integral polytope $P \subset M_{\mathbb{R}}$, then*

$$\varepsilon(X_t, L_t; 1) \geq s_1(P; P)$$

holds for very general $t \in T$.

Roughly speaking, this theorem states that we can obtain a lower bound of the Seshadri constant of (X, L) if (X, L) degenerates to a polarized toric variety.

By using Corollary 4.4, we obtain explicit estimations of Seshadri constants on hypersurfaces and Fano 3-folds with Picard number 1:

Theorem 1.8 (=Theorem 5.5). *Let X_d^n be a very general hypersurface of degree d in \mathbb{P}^{n+1} . Then it holds that*

$$\lfloor \sqrt[n]{d/(m_1^n + \dots + m_r^n)} \rfloor \leq \varepsilon(X_d^n, \mathcal{O}(1); \overline{m}) \leq \sqrt[n]{d/(m_1^n + \dots + m_r^n)}$$

for any $\overline{m} = (m_1, \dots, m_r) \in (\mathbb{N} \setminus 0)^r$.

In particular, it holds that

$$\lfloor \sqrt[n]{d} \rfloor \leq \varepsilon(X_d^n, \mathcal{O}(1); 1) \leq \sqrt[n]{d}.$$

Remark 1.9. Note that Theorem 1.8 does not hold for $\overline{m} \in \mathbb{R}_{>0}^r$ in general.

Theorem 1.10 (=Theorem 5.7). *For each family of smooth Fano 3-folds with Picard number 1 (note that there are 17 such families), $\varepsilon(X, -K_X; 1)$ is as in Table 1, where X is a very general member in the family.*

No.	Index	$(-K_X)^3$	Description	$\varepsilon(X, -K_X; 1)$
1	1	2	smooth hypersurface of degree 6 in $\mathbb{P}(1, 1, 1, 1, 3)$ (double cover of \mathbb{P}^3 ramified over smooth sextic)	6/5
2	1	4	the general element of the family is quartic	4/3
3	1	6	V_6 , smooth complete intersection of quadric and cubic	3/2
4	1	8	V_8 , smooth complete intersection of three quadrics	2
5	1	10	the general element is V_{10} , a section of $G(2, 5)$ by 2 hyperplanes in Plücker embedding and quadric	2
6	1	12	variety V_{12}	2
7	1	14	variety V_{14} , a section of $G(2, 6)$ by 5 hyperplanes in Plücker embedding	2
8	1	16	variety V_{16}	2
9	1	18	variety V_{18}	2
10	1	22	variety V_{22}	2
11	2	$8 \cdot 1$	smooth hypersurface of degree 6 in $\mathbb{P}(1, 1, 1, 2, 3)$ (double cover of the cone over the Veronese surface branched in a smooth cubic)	2
12	2	$8 \cdot 2$	smooth hypersurface of degree 4 in $\mathbb{P}(1, 1, 1, 1, 2)$ (double cover of \mathbb{P}^3 ramified over smooth quartic)	2
13	2	$8 \cdot 3$	smooth cubic	2
14	2	$8 \cdot 4$	smooth intersection of two quadrics	2
15	2	$8 \cdot 5$	variety V_5 , a section of $G(2, 5)$ by 3 hyperplanes in Plücker embedding	2
16	3	$27 \cdot 2$	smooth quadric	3
17	4	$64 \cdot 1$	\mathbb{P}^3	4

Table 1: Seshadri constants on Fano 3-folds with $\rho = 1$

This paper is organized as follows: In Section 2, we prepare some notations and conventions. In Section 3, we examine Seshadri constants on toric varieties and show Theorem 1.3. We also compute some examples. In Section 4, we prove Theorem 1.7. In Section 5, we verify Theorems 1.8 and Theorem 1.10.

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2. NOTATIONS AND CONVENTIONS

We denote by $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$, and \mathbb{C} the set of all natural numbers, integers, rational numbers, real numbers and complex numbers respectively. In this paper, \mathbb{N} contains 0. We define $\mathbb{R}_{\geq 0} := \{x \in \mathbb{R} \mid x \geq 0\}$ and $\mathbb{R}_{> 0} := \{x \in \mathbb{R} \mid x > 0\}$. For $x \in \mathbb{R}$, $\lfloor x \rfloor, \lceil x \rceil \in \mathbb{Z}$ are the round down and the round up of x respectively. We denote by e_1, \dots, e_n the standard basis of \mathbb{Z}^n or \mathbb{R}^n .

Unless otherwise stated, M stands for a free abelian group of rank $n \in \mathbb{N}$ in this paper. We define $M_K := M \otimes_{\mathbb{Z}} K$ for any field K . For a subset $S \subset M_{\mathbb{R}}$, we denote the convex hull of S by $\text{conv}(S)$. We write $\Sigma(S)$ for the closed convex cone S spans. For $t \in \mathbb{R}_{\geq 0}$, $tS := \{tu \mid u \in S\}$. For $u \in M_{\mathbb{R}}$, $S + u := \{u' + u \mid u' \in S\}$ is the parallel translation of S by u .

A subset $P \subset M_{\mathbb{R}}$ is called a polytope if it is the convex hull of a finite set in $M_{\mathbb{R}}$. A polytope P is integral (resp. rational) if all vertices are in M (resp. $M_{\mathbb{Q}}$). When σ is a face of a polytope $P \subset M_{\mathbb{R}}$, we write $\sigma \prec P$.

For a polytope $P \subset M_{\mathbb{R}}$, we denote by $\text{vol}_M(P)$ or $\text{vol}(P)$ the Euclidean volume of P under an identification of $M \subset M_{\mathbb{R}}$ with $\mathbb{Z}^n \subset \mathbb{R}^n$. Of course, $\text{vol}(P)$ does not depend on the identification. When $n = 1$, we write it $|P|_M$ or $|P|$, and call it the length of P . The dimension of P is the dimension of the affine space spanned by P .

For free abelian groups M and M' of rank n and r , a linear map $\pi : M_{\mathbb{R}} \rightarrow M'_{\mathbb{R}}$ is called a lattice projection if π is induced from a surjective group homomorphism $M \rightarrow M'$.

For a subset S in a topological space, we denote the closure of S by \bar{S} .

For a variety X , we say a property holds at a very general point of X if it holds for all points in the complement of the union of countably many proper subvarieties.

Throughout this paper, a divisor means a Cartier divisor. Hence a \mathbb{Q}, \mathbb{R} -divisor means a \mathbb{Q}, \mathbb{R} -Cartier \mathbb{Q}, \mathbb{R} -Weil divisor respectively. We use the words "divisor", "line bundle", and "invertible sheaf" interchangeably.

We call a pair (X, L) a (\mathbb{Q}) -polarized variety if X is a projective variety and L is an ample (\mathbb{Q}) -line bundle on X . The normalization of a \mathbb{Q} -polarized variety (X, L) is (X^{nor}, π^*L) , where $\pi : X^{nor} \rightarrow X$ is the normalization of X .

3. SESHADRI CONSTANTS ON TORIC VARIETIES

In this section, we investigate Seshadri constants on toric varieties and prove Theorem 1.3. We refer the reader to [Fu] for toric varieties.

Definition 3.1. Let $\Gamma \subset \mathbb{N} \times M$ be a finitely generated subsemigroup such that $\Gamma \cap (\{0\} \times M) = \{0\}$ and Γ generates $\mathbb{Z} \times M$ as a group. We define a not necessarily normal \mathbb{Q} -polarized toric variety $(X(\Gamma), L(\Gamma))$ as follows:

$$(X(\Gamma), L(\Gamma)) := (\text{Proj } \mathbb{C}[\Gamma], \mathcal{O}_{\text{Proj } \mathbb{C}[\Gamma]}(1)).$$

Note that the torus $T_M := \text{Spec } \mathbb{C}[M]$ naturally acts on $(X(\Gamma), L(\Gamma))$.

The moment polytope $\Delta(\Gamma)$ of $(X(\Gamma), L(\Gamma))$ is defined to be

$$\Delta(\Gamma) := \Sigma(\Gamma) \cap (\{1\} \times M_{\mathbb{R}}) \subset \{1\} \times M_{\mathbb{R}},$$

which can be regarded as a rational polytope in $M_{\mathbb{R}}$ naturally.

For a rational polytope $P \subset M_{\mathbb{R}}$ of dimension n , we define the normal \mathbb{Q} -polarized toric variety (X_P, L_P) by

$$(X_P, L_P) := (X(\Gamma_P), L(\Gamma_P)),$$

where $\Gamma_P := \Sigma(\{1\} \times P) \cap (\mathbb{N} \times M)$. We write the maximal orbit of X_P as O_P , and denote by $1_P \in O_P = T_M$ the identity of the torus. For a face σ of P , there is a natural closed embedding $X_{\sigma} \hookrightarrow X_P$. Hence we can regard X_{σ} as a closed subvariety of X_P , and O_{σ} is considered as a T_M -orbit in X_P .

Remark 3.2. For any Γ , the normalization of $(X(\Gamma), L(\Gamma))$ is $\mu : (X_{\Delta(\Gamma)}, L_{\Delta(\Gamma)}) \rightarrow (X(\Gamma), L(\Gamma))$ induced by $\Gamma \hookrightarrow \Sigma(\Gamma) \cap (\mathbb{N} \times M)$ (cf. [Ei, Exercise 4.22]). When P is an integral polytope, L_P is a line bundle.

Remark 3.3. For any integral polytope $P \subset M_{\mathbb{R}}$ of dimension n and any face $\sigma \prec P$, $\varepsilon(X_P, L_P; p)$ is constant for $p \in O_{\sigma}$ because of the torus action. In particular, $\varepsilon(X_P, L_P; 1)$ (in the sense of Definition 1.4) coincides with $\varepsilon(X_P, L_P; 1_P)$.

3.1. At a point in the maximal orbit. In this subsection, we estimate $\varepsilon(X_P, L_P; 1_P)$ for an integral polytope P .

Lemma 3.4. *Let $\pi : M_{\mathbb{R}} \rightarrow M'_{\mathbb{R}}$ be a lattice projection with $\text{rank } M = n, \text{rank } M' = r$, $P \subset M_{\mathbb{R}}$ an integral polytope of dimension n .*

Then the closure $(\overline{T_{M'}}, L_P|_{\overline{T_{M'}}})$ of $T_{M'}$ in X_P is a not necessarily normal polarized toric variety whose moment polytope is $\pi(P) \subset M'_{\mathbb{R}}$, where $T_{M'} \hookrightarrow T_M = O_P \subset X_P$ is induced by the surjection $\pi|_M : M \rightarrow M'$.

Proof. For simplicity, we set $P' = \pi(P)$.

There is a commutative diagram

$$\begin{array}{ccc} \Sigma(\{1\} \times P) & \hookrightarrow & \mathbb{R} \times M_{\mathbb{R}} \\ \downarrow & \circlearrowleft & \downarrow \text{id}_{\mathbb{R}} \times \pi \\ \Sigma(\{1\} \times P') & \hookrightarrow & \mathbb{R} \times M'_{\mathbb{R}}. \end{array}$$

By intersecting with $\mathbb{N} \times M$ or $\mathbb{N} \times M'$, we have

$$\begin{array}{ccc} \Gamma_P = \Sigma(\{1\} \times P) \cap (\mathbb{N} \times M) & \hookrightarrow & \mathbb{N} \times M \\ \downarrow & \circlearrowleft & \downarrow \text{id}_{\mathbb{N}} \times \pi|_M \\ \Gamma_{P'} = \Sigma(\{1\} \times P') \cap (\mathbb{N} \times M') & \hookrightarrow & \mathbb{N} \times M'. \end{array}$$

Note that the above $\Gamma_P \rightarrow \Gamma_{P'}$ is not necessarily surjective. Set $\Gamma' = (\text{id}_{\mathbb{N}} \times \pi|_M)(\Gamma_P)$. Then Γ' generates $\mathbb{Z} \times M'$ as a group and $\Delta(\Gamma') = P'$ because $\dim P = n$ and $\pi|_M : M \rightarrow M'$ is surjective.

The above diagram induces

$$\begin{array}{ccc}
 X_P = \text{Proj } \mathbb{C}[\Gamma_P] & \xleftarrow{i} & T_M \\
 \uparrow \iota & \circlearrowleft & \uparrow \\
 X(\Gamma') = \text{Proj } \mathbb{C}[\Gamma'] & \xleftarrow{i'} & T_{M'} \\
 \uparrow \nu & & \parallel \\
 X_{P'} = \text{Proj } \mathbb{C}[\Gamma_{P'}] & \xleftarrow{i'} & T_{M'},
 \end{array}$$

where ι is a closed embedding, i and i' are open immersions, and ν is the normalization. Therefore $(\overline{T_{M'}}, L_P|_{\overline{T_{M'}}}) = (X(\Gamma'), L(\Gamma'))$, whose moment polytope is P' . \square

Let $\pi : M_{\mathbb{R}} \rightarrow M'_{\mathbb{R}}, P$, and $P' = \pi(P)$ be as in Lemma 3.4, and set

$$P(u') = \pi^{-1}(u') \cap P$$

for $u' \in P' \cap M'_{\mathbb{Q}}$. An splitting $M \cong \ker \pi|_M \oplus M'$ of $0 \rightarrow \ker \pi|_M \rightarrow M \xrightarrow{\pi|_M} M' \rightarrow 0$ induces the identification of $\pi^{-1}(u')$ with $\ker \pi$, hence we can consider $P(u')$ as a rational polytope in $\ker \pi = (\ker \pi|_M)_{\mathbb{R}}$. Assume that the dimension of $P(u')$ is $n - r$. Then $P(u')$ defines the \mathbb{Q} -polarized toric variety $(X_{P(u')}, L_{P(u')})$, and the isomorph class of $(X_{P(u')}, L_{P(u')})$ does not depend on the choice of $M \cong \ker \pi|_M \oplus M'$.

Lemma 3.5. *Let $\pi : M_{\mathbb{R}} \rightarrow M'_{\mathbb{R}}, P$, and $P' = \pi(P)$ be as in Lemma 3.4, and take $u' \in P' \cap M'_{\mathbb{Q}}$ such that $\dim P(u') = n - r$. Then, there exists a generically surjective rational map $\varphi : X_P \dashrightarrow X_{P(u')}$ such that for any resolution $\mu : Y \rightarrow X_P$ of the indeterminacy of φ , the following conditions hold:*

- (i) $\mu^* L_P - f^* L_{P(u')}$ is \mathbb{Q} -effective, where $f = \varphi \circ \mu$,
- (ii) $\mu(f^{-1}(1_{P(u')})) \cap O_P = T_{M'}$ holds for $1_{P(u')} \in O_{P(u')} \subset X_{P(u')}$.

$$\begin{array}{ccc}
 Y & \xrightarrow{\mu} & X_P \\
 & \searrow f & \downarrow \varphi \\
 & & X_{P(u')}
 \end{array}$$

Proof. By considering kP for sufficiently large and divisible $k \in \mathbb{N}$, we may assume u' is contained in M' and $P(u')$ is an integral polytope. Furthermore, by considering $P - u$ for $u \in (\pi|_M)^{-1}(u')$, we may assume $u' = 0 \in M'$. Hence $P(u')$ is an integral polytope in $\pi^{-1}(0) = (\ker \pi|_M)_{\mathbb{R}}$.

There is a commutative diagram

$$\begin{array}{ccc}
 \Sigma(\{1\} \times P) & \hookrightarrow & \mathbb{R} \times M_{\mathbb{R}} \\
 \uparrow & \circlearrowleft & \uparrow \\
 \Sigma(\{1\} \times P(u')) & \hookrightarrow & \mathbb{R} \times \ker \pi.
 \end{array}$$

By intersecting with $\mathbb{N} \times M$ or $\mathbb{N} \times \ker \pi|_M$, we have

$$\begin{array}{ccc} \Gamma_P = \Sigma(\{1\} \times P) \cap (\mathbb{N} \times M) & \hookrightarrow & \mathbb{N} \times M \\ \uparrow \psi & \circlearrowleft & \uparrow \\ \Gamma_{P(u')} = \Sigma(\{1\} \times P(u')) \cap (\mathbb{N} \times \ker \pi|_M) & \hookrightarrow & \mathbb{N} \times \ker \pi|_M. \end{array}$$

This diagram induces

$$\begin{array}{ccc} X_P = \text{Proj } \mathbb{C}[\Gamma_P] & \longleftarrow \longrightarrow & O_P = T_M \\ \downarrow \varphi & \circlearrowleft & \downarrow \varphi|_{T_M} \\ X_{P(u')} = \text{Proj } \mathbb{C}[\Gamma_{P(u')}] & \longleftarrow \longrightarrow & O_{P(u')} = T_{\ker \pi|_M}. \end{array}$$

Then φ is generically surjective because $\varphi|_{T_M}$ is surjective. We show this φ satisfies (i) and (ii) in the statement of this lemma.

Clearly $\varphi|_{T_M}^{-1}(1_{P(u')}) = T_{M'}$, hence (ii) holds.

Let $\mu : Y \rightarrow X_P$ be a resolution of indeterminacy of φ . Then $X_P, X_{P(u')}$ are normal and μ, f have connected fibers. Thus

$$\begin{aligned} \bigoplus_{k \in \mathbb{N}} H^0(Y, k f^* L_{P(u')}) &= \bigoplus_{k \in \mathbb{N}} H^0(X_{P(u')}, k L_{P(u')}) = \Gamma_{P(u')}, \\ \bigoplus_{k \in \mathbb{N}} H^0(Y, k \mu^* L_P) &= \bigoplus_{k \in \mathbb{N}} H^0(X_P, k L_P) = \Gamma_P. \end{aligned}$$

Therefore an injection $f^* L_{P(u')} \hookrightarrow \mu^* L_P$ is induced from the injection

$$\bigoplus_{k \in \mathbb{N}} H^0(Y, k f^* L_{P(u')}) = \Gamma_{P(u')} \xrightarrow{\psi} \Gamma_P = \bigoplus_{k \in \mathbb{N}} H^0(Y, k \mu^* L_P).$$

Hence (i) holds. \square

We need one more lemma, which states that lower and upper bounds of Seshadri constants are obtained from surjective morphisms.

Lemma 3.6. *Let $f : Y \rightarrow Z$ be a surjective morphism between projective varieties. Assume that L, L' are nef and big \mathbb{Q} -divisors on Y, Z respectively such that $L - f^* L'$ is \mathbb{Q} -effective. Set $\mathbf{B}(L - f^* L') = Bs(|k(L - f^* L')|)$ for sufficiently large and divisible $k \in \mathbb{N}$ (which is called the stable base locus of $L - f^* L'$, and does not depend on k). See [La2, Remark 2.1.24]. Then*

$$\min\left\{\min_{1 \leq i \leq r} \varepsilon(Y_i, L|_{Y_i}; y), \varepsilon(Z, L'; f(y))\right\} \leq \varepsilon(Y, L; y) \leq \min_{1 \leq i \leq r} \varepsilon(Y_i, L|_{Y_i}; y)$$

holds for $y \notin \mathbf{B}(L - f^* L')$, where Y_1, \dots, Y_r are all the irreducible components of $f^{-1}(f(y))$ containing y with the reduced structures.

Proof. We may assume L and L' are ample. In fact, for nef and big L, L' , choose ample divisors A, A' on Y, Z such that $y \notin \mathbf{B}(A - f^* A')$, and consider $L + \delta A, L' + \delta A' (\delta > 0)$ instead of L, L' . Then we can show this lemma from ample cases by $\delta \rightarrow 0$.

The second inequality is clear by the definition of Seshadri constants, thus it is enough to show the first one. For the sake of simplicity, we set $z = f(y)$.

Fix a curve $C \subset Y$ containing y .

It suffices to show $\min\{\min_i \varepsilon(Y_i, L|_{Y_i}; y), \varepsilon(Z, L'; f(y))\} \leq \frac{C.L}{\text{mult}_y(C)} \cdots (*)$

Case 1. $C \subset f^{-1}(z)$.

Since $C \subset Y_i$ for some i , $\frac{C.L}{\text{mult}_y(C)} = \frac{C.L|_{Y_i}}{\text{mult}_y(C)} \geq \varepsilon(Y_i, L|_{Y_i}; y)$ holds.

Case 2. $C \not\subset f^{-1}(z)$.

Set $C' = f(C)$ with the reduced structure and fix a rational number $0 < t < \varepsilon(Z, L'; z)$. Then for any sufficiently large and divisible $k \in \mathbb{N}$, there exists $D' \in |kL' \otimes \mathfrak{m}_z^{kt}|$ such that $C' \not\subset \text{Supp } D'$ by the ampleness of L' and [La2, Lemma 5.4.24]. Clearly $f^*D' \in |kf^*L' \otimes \mathfrak{m}_y^{kt}|$ and $C \not\subset \text{Supp } f^*D'$, hence we have

$$\begin{aligned} k(C.L) &= kC.(L - f^*L' + f^*L') \\ &= kC.(L - f^*L') + C.f^*D' \\ &\geq C.f^*D' \\ &\geq kt \cdot \text{mult}_y(C). \end{aligned}$$

Note $C.(L - f^*L') \geq 0$ holds by the assumption $y \notin \mathbf{B}(L - f^*L')$. Therefore $\frac{C.L}{\text{mult}_y(C)} \geq t$

holds and we have $\frac{C.L}{\text{mult}_y(C)} \geq \varepsilon(Z, L'; z)$ by $t \rightarrow \varepsilon(Z, L'; z)$.

Thus for any curve $C \subset Y$ containing y , $(*)$ holds. \square

By Lemmas 3.4, 3.5, and 3.6, we obtain the following proposition, which is useful to estimate $\varepsilon(X_P, L_P; 1_P)$. The lower bound in this proposition is essentially a generalization of an Eckl's result [Ec, Theorem 2.2], which is the case $n = 2, r = 1$. Note that Eckl's proof is rather algebraic in comparison with our geometrical proof.

Proposition 3.7. *Let $\pi : M_{\mathbb{R}} \rightarrow M'_{\mathbb{R}}$ be a lattice projection for free abelian groups M and M' of rank n and r . Let $P \subset M_{\mathbb{R}}$ be an n -dimensional integral polytope, and set $P' = \pi(P)$. Fix $u' \in P' \cap M'_{\mathbb{Q}}$ such that $\dim P(u') = n - r$. Then it holds that*

$$\min\{\varepsilon(X_{P'}, L_{P'}; 1_{P'}), \varepsilon(X_{P(u')}, L_{P(u')}; 1_{P(u')})\} \leq \varepsilon(X_P, L_P; 1_P) \leq \varepsilon(X_{P'}, L_{P'}; 1_{P'}).$$

Proof. Let $\varphi : X_P \dashrightarrow X_{P(u')}$ be the rational map defined in Lemma 3.5. For a toric resolution $\mu : Y \rightarrow X_P$ of the indeterminacy of φ , the stable base locus $\mathbf{B}(\mu^*L_P - f^*L_{P(u')})$ is contained in $Y \setminus O$, where O is the maximal orbit of Y . By applying Lemma 3.6 to $f : Y \rightarrow X_{P(u')}, \mu^*L_P, L_{P(u')}$ and the identity 1_Y of the torus $O \subset Y$, we have

$$\begin{aligned} \min\{\varepsilon(Y_1, (\mu^*L_P)|_{Y_1}; 1_Y), \varepsilon(X_{P(u')}, L_{P(u')}; 1_{P(u')})\} \\ \leq \varepsilon(Y, \mu^*L_P; 1_Y) \leq \varepsilon(Y_1, (\mu^*L_P)|_{Y_1}; 1_Y), \end{aligned}$$

where Y_1 is the irreducible component of $f^{-1}(1_{P(u')})$ containing 1_Y . Since $\mu : Y \rightarrow X_P$ and $f|_{Y_1} : Y_1 \rightarrow \overline{T_{M'}}(\subset X_P)$ are birational and isomorphic around 1_Y from the proof of Lemma 3.5, it holds that $\varepsilon(Y, \mu^*L_P; 1_Y) = \varepsilon(X_P, L_P; 1_P)$, $\varepsilon(Y_1, (\mu^*L_P)|_{Y_1}; 1_Y) = \varepsilon(\overline{T_{M'}}, L_P|_{\overline{T_{M'}}}; 1_P)$. The normalization of $(\overline{T_{M'}}, L_P|_{\overline{T_{M'}}})$ is $(X_{P'}, L_{P'})$ by Lemma 3.4. Thus we have

$$\varepsilon(Y_1, (\mu^*L_P)|_{Y_1}; 1_Y) = \varepsilon(\overline{T_{M'}}, L_P|_{\overline{T_{M'}}}; 1_P) = \varepsilon(X_{P'}, L_{P'}; 1_{P'}).$$

From these equalities, this proposition follows. \square

In view of Proposition 3.7, we define invariants $s_1(P)$ and $s_2(P)$ for a rational polytope $P \subset \mathbb{R}^n$ as follows:

Definition 3.8. Let P be a rational polytope in $M_{\mathbb{R}}$. We define $s_1(P) = s_1^M(P) \in \mathbb{R}_{\geq 0}$ for which $s_1(P + u) = s_1(P)$ holds for any $u \in M_{\mathbb{Q}}$ by induction of n as follows:

When $n = 1$, we define $s_1(P) = |P|_M$, the length of P . Note that $M \subset M_{\mathbb{R}}$ is identified with $\mathbb{Z} \subset \mathbb{R}$ as stated in Section 2. Clearly $s_1(P + u) = s_1(P)$ holds for any $u \in M_{\mathbb{Q}}$.

Assume such $s_1(P)$ is defined in the case of rank $n - 1$, and set

$$\Phi = \{ \pi : M_{\mathbb{R}} \rightarrow (\mathbb{Z})_{\mathbb{R}} = \mathbb{R} \mid \pi \text{ is a lattice projection} \}.$$

Fix $\pi \in \Phi$ and choose a splitting $M \cong \ker \pi|_M \oplus \mathbb{Z}$ of $0 \rightarrow \ker \pi|_M \rightarrow M \xrightarrow{\pi|_M} \mathbb{Z} \rightarrow 0$. Then for $u' \in \mathbb{Q}$, $\pi^{-1}(u') \cap P$ can be regarded as a rational polytope in $\ker \pi = (\ker \pi|_M)_{\mathbb{R}}$ naturally. Thus we can define $s_1^{\ker \pi|_M}(\pi^{-1}(u') \cap P) \in \mathbb{R}_{\geq 0}$ by the induction hypothesis. Another choice of the splitting only causes a parallel translation of $\pi^{-1}(u') \cap P$ in $\ker \pi$, hence $s_1^{\ker \pi|_M}(\pi^{-1}(u') \cap P)$ does not depend on the splitting by the induction hypothesis. We define

$$s_1(P) = s_1^M(P) := \sup_{\pi \in \Phi} \min \{ |\pi(P)|_{\mathbb{Z}}, \sup_{u' \in \mathbb{Q}} s_1^{\ker \pi|_M}(\pi^{-1}(u') \cap P) \}.$$

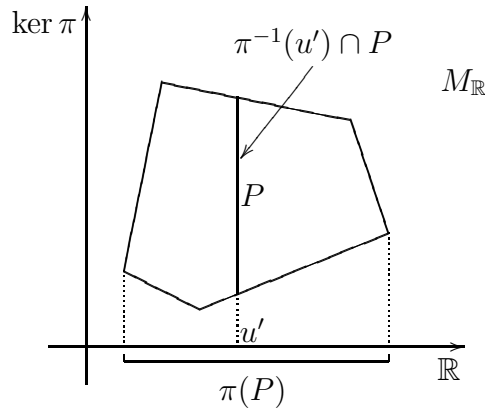
Clearly, $s_1(P + u) = s_1(P)$ holds for any $u \in M_{\mathbb{Q}}$.

The definition of $s_2(P)$ is more simple. For a rational polytope $P \subset M_{\mathbb{R}}$, $s_2(P) \in \mathbb{R}_{\geq 0}$ is defined to be

$$s_2(P) = \inf_{\pi \in \Phi} |\pi(P)|_{\mathbb{Z}}.$$

By definition, $s_2(P + u) = s_2(P)$ holds for any P and $u \in M_{\mathbb{Q}}$.

We define $s_1(\{0\}) = s_2(\{0\}) = +\infty$, $s_1(\emptyset) = s_2(\emptyset) = 0$ if $n = 0$.



Remark 3.9. Let V be a finite dimensional \mathbb{R} -vector space, and take two lattices M_1, M_2 of V , hence $(M_1)_{\mathbb{R}} = (M_2)_{\mathbb{R}} = V$. In general $s_i^{M_1}(P) \neq s_i^{M_2}(P)$ for $P \subset V$ and $i = 1, 2$. Thus we have to notice which lattice we consider about when we deal with $s_1(\cdot), s_2(\cdot)$.

By Proposition 3.7, we can show that $s_1(P)$ and $s_2(P)$ give a lower bound and an upper bound of $\varepsilon(X_P, L_P; 1_P)$ respectively:

Proposition 3.10. *Let $P \subset M_{\mathbb{R}}$ be a rational polytope of dimension n . Then for any $p \in O_P \subset X_P$, it holds that*

$$s_1(P) \leq \varepsilon(X_P, L_P; p) \leq s_2(P).$$

Proof. By the torus action, we may assume $p = 1_P$. We show this proposition by induction of n .

If $n = 1$, then $\varepsilon(X_P, L_P; 1_P) = \deg(L_P) = |P|$. By definitions $s_1(P) = s_2(P) = |P|$, thus the inequalities in the proposition follow.

We assume this proposition holds if the rank of M is $n - 1$, and show the case of rank n .

We use the notations in Definition 3.8. Fix $\pi \in \Phi$, i.e., $\pi : M_{\mathbb{R}} \rightarrow \mathbb{R} = (\mathbb{Z})_{\mathbb{R}}$ is a lattice projection. We can apply Proposition 3.7 to P and $u' \in \pi(P) \cap \mathbb{Q}$ such that $\dim(\pi^{-1}(u') \cap P) = n - 1$. Then we obtain inequalities

$$\begin{aligned} \min\{\varepsilon(X_{\pi(P)}, L_{\pi(P)}; 1_{\pi(P)}), \varepsilon(X_{\pi^{-1}(u') \cap P}, L_{\pi^{-1}(u') \cap P}; 1_{\pi^{-1}(u') \cap P})\} \\ \leq \varepsilon(X_P, L_P; 1_P) \leq \varepsilon(X_{\pi(P)}, L_{\pi(P)}; 1_{\pi(P)}). \end{aligned}$$

Note that Proposition 3.7 can be applied to rational polytopes. Now $\varepsilon(X_{\pi(P)}, L_{\pi(P)}; 1_{\pi(P)}) = |\pi(P)|$, and by the induction hypothesis we have

$$s_1(\pi^{-1}(u') \cap P) \leq \varepsilon(X_{\pi^{-1}(u') \cap P}, L_{\pi^{-1}(u') \cap P}; 1_{\pi^{-1}(u') \cap P}).$$

Thus these inequalities induce

$$\min\{|\pi(P)|, s_1(\pi^{-1}(u') \cap P)\} \leq \varepsilon(X_P, L_P; 1_P) \leq |\pi(P)| \cdots (*)$$

Note that $(*)$ also holds if $\dim(\pi^{-1}(u') \cap P) < n - 1$ since $s_1(\pi^{-1}(u') \cap P) = 0$ for such $u' \in \mathbb{Q}$. (This can be shown easily by the definition of s_1 .) Moving u' , we have

$$\min\{|\pi(P)|, \sup_{u' \in \mathbb{Q}} s_1(\pi^{-1}(u') \cap P)\} \leq \varepsilon(X_P, L_P; 1_P) \leq |\pi(P)|.$$

By moving π , we obtain $s_1(P) \leq \varepsilon(X_P, L_P; 1_P) \leq s_2(P)$. \square

Remark 3.11. (1) Note that $s_2(P)$ is called the lattice width of P . The author [It2] proved that $\varepsilon(X_P, L_P; 1_P) = 1$ if and only if $s_2(P) = 1$ for any integral polytope $P \subset M_{\mathbb{R}}$ of dimension n . But in general, $\varepsilon(X_P, L_P; 1_P) \neq s_2(P)$. See Example 3.12 (3).

(2) If $|\pi(P)| \leq s_1(\pi^{-1}(u') \cap P)$ holds for some $\pi \in \Phi$ and $u' \in \mathbb{Q}$, we have $\varepsilon(X_P, L_P; 1_P) = |\pi(P)| = s_1(P) = s_2(P)$ by Proposition 3.10.

(3) The upper bound $s_2(P)$ can be a little improved. In fact,

$$\varepsilon(X_P, L_P; 1_P) \leq \inf_{\pi: M_{\mathbb{R}} \rightarrow M'_{\mathbb{R}}} \sqrt[\text{rank } M']{(\text{rank } M')! \text{vol}_{M'_{\mathbb{R}}}(\pi(P))}$$

holds, where $\pi : M_{\mathbb{R}} \rightarrow M'_{\mathbb{R}}$ moves all lattice projections from $M_{\mathbb{R}}$. This is shown from Proposition 3.7 and Remark 1.2 immediately.

3.2. At a point in the maximal orbit, Examples. By using Propositions 3.7, 3.10, we estimate $\varepsilon(X_P, L_P; 1_P)$ for some P .

Example 3.12. (1) Set $P_n = \text{conv}(0, e_1, \dots, e_n)$ for the standard basis e_1, \dots, e_n of \mathbb{Z}^n . We apply Proposition 3.7 to the n -th projection $\pi : \mathbb{R}^n \rightarrow \mathbb{R}$ and $u' = 0 \in \mathbb{R}$. Since $P' = \pi(P) = [0, 1] \subset \mathbb{R}$, we have $\varepsilon(X_{P'}, L_{P'}; 1_{P'}) = |P'| = 1$. On the other hand, $P(u') = P_{n-1} = \text{conv}(0, e_1, \dots, e_{n-1}) \subset \mathbb{R}^{n-1}$. By Proposition 3.7, it holds that

$$\min\{1, \varepsilon(X_{P_{n-1}}, L_{P_{n-1}}; 1_{P_{n-1}})\} \leq \varepsilon(X_{P_n}, L_{P_n}; 1_{P_n}) \leq 1.$$

Since $\varepsilon(X_{P_1}, L_{P_1}; 1_{P_1}) = |P_1| = 1$, we have $\varepsilon(X_{P_n}, L_{P_n}; 1_{P_n}) = 1$ for any n inductively. Note that $(X_{P_n}, L_{P_n}) = (\mathbb{P}^n, \mathcal{O}(1))$.

(2) Set $P = \text{conv}((0, 0), (a, 0), (0, b), (a, b)) \subset \mathbb{R}^2$ for $a \leq b$ in $\mathbb{N} \setminus 0$. We apply Proposition 3.7 to the first projection $\pi : \mathbb{R}^2 \rightarrow \mathbb{R}$ and $u' = 0 \in \mathbb{R}$. Then $\varepsilon(X_P, L_P; 1_P) = \varepsilon(X_{P'}, L_{P'}; 1_{P'}) = a$ by Remark 3.11 (2) because $|P'| = a \leq b = s_1(\pi^{-1}(0) \cap P)$. Note that $(X_P, L_P) = (\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(a, b))$.

(3) Set $P = \text{conv}((1, 0), (0, 1), (-1, -1)) \subset \mathbb{R}^2$. Since P is a triangle, it holds $|\pi^{-1}(u') \cap P| \cdot \pi(P) = 2 \cdot \text{vol}(P) = 3$ for any lattice projection $\pi : \mathbb{R}^2 \rightarrow \mathbb{R}$, where $\pi^{-1}(u') \cap P$ is the longest fiber of $P \rightarrow \pi(P)$. Thus $s_1(P) = \min\{s_2(P), 3/s_2(P)\}$ holds. It is easy to see $s_2(P) = 2$, hence we have $s_1(P) = 3/s_2(P) = 3/2$. Thus $3/2 \leq \varepsilon(X_P, L_P; 1_P) \leq 2$ holds by Proposition 3.10. Note that X_P is the singular cubic surface in $\mathbb{P}^3 = \text{Proj } \mathbb{C}[T_0, T_1, T_2, T_3]$ defined by $T_0^3 = T_1 T_2 T_3$ and $L_P = \mathcal{O}(1)$. For any (integral and not necessarily smooth) cubic surface $S \subset \mathbb{P}^3$ and a general point $p \in S$, the plane in \mathbb{P}^3 tangent to S at p induces a singular curve $C \sim \mathcal{O}_S(1)$. Thus $\varepsilon(S, \mathcal{O}(1); p) \leq 3/2$ holds. Hence we have $s_1(P) = 3/2 = \varepsilon(X_P, L_P; 1_P) < s_2(P)$ in this case.

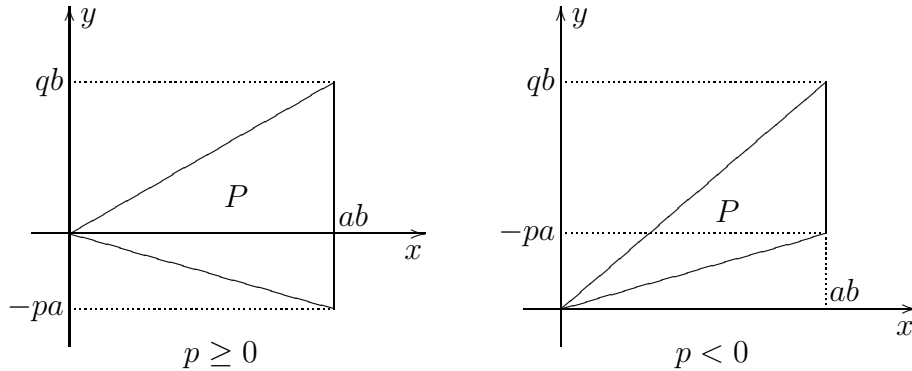
(4) It is well known that there are five toric Del Pezzo surfaces. For an integral polytope $P \subset \mathbb{R}^2$ such that X_P is a Del pezzo surface and $L_P = -K_{X_P}$, we can easily find a projection π and $u' \in \mathbb{Q}$ as in Remark 3.11 (2) and compute $\varepsilon(X_P, L_P; 1_P)$. As a consequence, we have

$$\varepsilon(X_P, L_P; 1_P) = \begin{cases} 3 & \text{if } X_P = \mathbb{P}^2 \\ 2 & \text{otherwise} \end{cases}$$

for such P .

In the above examples, Seshadri constants can be computed without using Propositions 3.7, 3.10. The following examples are new computations of Seshadri constants on toric varieties.

(5) We consider a weighted projective space $\mathbb{P}(a, b, c)$ with $c = \max\{a, b, c\}$. We may assume any two of a, b, c are coprime. Since a and b are coprime, we can denote $c = pa + qb$ for integers p, q such that $0 \leq q < a$. Let $P \subset \mathbb{R}^2$ be the convex hull of $(0, 0)$, (ab, qb) and $(ab, -pa)$.



It is easy to see that $(X_P, L_P) = (\mathbb{P}(a, b, c), \mathcal{O}(abc))$. Since P is a triangle, we have $s_1(P) = \min\{s_2(P), abc/s_2(P)\} \leq \varepsilon(X_P, L_P; 1_P) \leq s_2(P)$ as (3). In other words, it holds that

$$\min\{s_2(P)/abc, 1/s_2(P)\} \leq \varepsilon(\mathbb{P}(a, b, c), \mathcal{O}(1); 1) \leq s_2(P)/abc.$$

Since $s_2(P)$ can be computed by finite calculations for any given a, b, c (or more generally, any given integral polytope in \mathbb{R}^n), we obtain an explicit estimation. If $s_2(P) \leq \sqrt{abc}$, it holds $\varepsilon(\mathbb{P}(a, b, c), \mathcal{O}(1); 1) = s_2(P)/abc$. For example,

(i) When $p \geq 0$, we consider the first or second projections $\mathbb{R}^2 \rightarrow \mathbb{R}$ as π . Then we have $|\pi(P)| = \min\{ab, c\} \leq \sqrt{abc}$. Thus it holds

$$\varepsilon(\mathbb{P}(a, b, c), \mathcal{O}(1); 1) = \min\{ab, c\}/abc = \min\{1/c, 1/ab\}$$

by Remark 3.11 (2). For instance, $p \geq 0$ holds if $a = 1, 2$, or $ab \leq c$.

(ii) When $p < 0$, we have $|p_2(P)| = qb$ for the second projection $p_2 : \mathbb{R}^2 \rightarrow \mathbb{R}$. Thus, if $qb \leq \sqrt{abc}$, i.e., $q^2b \leq ac$, it holds that $\varepsilon(\mathbb{P}(a, b, c), \mathcal{O}(1); 1) = qb/abc = q/ac$. For example,

$$\varepsilon(\mathbb{P}(3, 5, 7), \mathcal{O}(1); 1) = 2/21$$

holds since $7 = -1 \cdot 3 + 2 \cdot 5$.

(iii) If $a = 3, b = 4, c = 5$, we have $s_2(P) = 8$. In this case, $s_2(P) = 8 > 2\sqrt{15} = \sqrt{abc}$. Thus we have only the estimation

$$1/8 \leq \varepsilon(\mathbb{P}(3, 4, 5), \mathcal{O}(1); 1) \leq 2/15.$$

(6) There are 18 smooth toric Fano 3-folds (cf. [Bat], [WW]). As (4), we can easily compute $\varepsilon(X_P, L_P; 1_P)$ if X_P is a smooth toric Fano 3-fold and $L_P = -K_{X_P}$. For such P , we can show

$$\varepsilon(X_P, L_P; 1_P) = \begin{cases} 4 & \text{if } X_P = \mathbb{P}^3 \\ 3 & \text{if } X_P = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)) \\ 2 & \text{otherwise.} \end{cases}$$

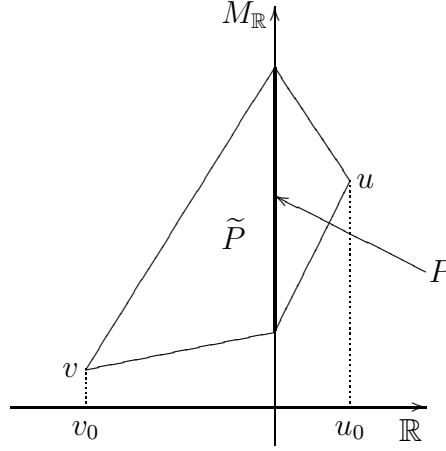
(7) We give examples of a polarized variety (X, L) satisfying

$$\varepsilon(X, L; 1) = \sqrt[n]{L^n} \in \mathbb{N} \cdots (*)$$

for $n = \dim X$. We construct such examples by induction of n as follows:

When $n = 1$, (X_P, L_P) satisfies the condition $(*)$ for any integral polytope P in $M_{\mathbb{R}} \cong \mathbb{R}$. Note $(*)$ always holds if X is a curve.

Let $P \subset M_{\mathbb{R}}$ be an integral polytope such that (X_P, L_P) satisfies $(*)$. Choose $u, v \in \mathbb{Z} \times M$ such that $\overline{uv} \cap (\{0\} \times P) \neq \emptyset$ and $|u_0 - v_0| = \sqrt[n]{L_P^n} =: m \in \mathbb{N}$, where \overline{uv} is the segment in $(\mathbb{Z} \times M)_{\mathbb{R}}$ whose end points are u and v , and u_0, v_0 are \mathbb{Z} -components of $u, v \in \mathbb{Z} \times M$ respectively. Set $\tilde{P} = \text{conv}(u, v, \{0\} \times P)$ in $(\mathbb{Z} \times M)_{\mathbb{R}}$. By applying Proposition 3.7 to the first projection $\pi : \mathbb{R} \times M_{\mathbb{R}} \rightarrow \mathbb{R}$ and $0 \in \mathbb{R}$, we have $\varepsilon(X_{\tilde{P}}, L_{\tilde{P}}; 1_{\tilde{P}}) = m$. Since $L_{\tilde{P}}^{n+1} = (n+1)! \text{vol}(\tilde{P}) = m^{n+1}$, $(X_{\tilde{P}}, L_{\tilde{P}})$ is an $n+1$ -dimensional example satisfying $(*)$.



(8) Set $P = \text{conv}(e_1, \dots, e_n, -\sum_{i=1}^n a_i e_i) \subset \mathbb{R}^n$ for rational numbers $a_1, \dots, a_n \geq 0$. Then we have

$$\varepsilon(X_P, L_P; 1_P) \geq \min_{1 \leq i \leq n} \frac{a_i + \dots + a_n + 1}{a_{i+1} + \dots + a_n + 1} \dots (**)$$

We show this by induction of n . When $n = 1$, $\varepsilon(X_P, L_P; 1_P) = |P| = a_1 + 1 = \min_{1 \leq i \leq n} \frac{a_i + \dots + a_n + 1}{a_{i+1} + \dots + a_n + 1}$. Thus $(**)$ holds.

Assume $(**)$ holds for $n-1$. We apply Proposition 3.7 to the n -th projection $\pi : \mathbb{R}^n \rightarrow \mathbb{R}$, P and $0 \in \pi(P) \cap \mathbb{Q}$. Then $P(0) = \pi^{-1}(0) \cap P = \text{conv}(e_1, \dots, e_{n-1}, -1/(a_n + 1) \sum_{i=1}^{n-1} a_i e_i)$ and $P' = \pi(P) = [-a_n, 1] \subset \mathbb{R}$. By induction hypothesis,

$$\begin{aligned} \varepsilon(X_{P(0)}, L_{P(0)}; 1_{P(0)}) &\geq \min_{1 \leq i \leq n-1} \frac{a_i/(a_n + 1) + \dots + a_{n-1}/(a_n + 1) + 1}{a_{i+1}/(a_n + 1) + \dots + a_{n-1}/(a_n + 1) + 1} \\ &= \min_{1 \leq i \leq n-1} \frac{a_i + \dots + a_n + 1}{a_{i+1} + \dots + a_n + 1} \end{aligned}$$

holds. By Proposition 3.7, it follows that

$$\begin{aligned} \varepsilon(X_P, L_P; 1_P) &\geq \min\{\varepsilon(X_{P'}, L_{P'}; 1_{P'}), \varepsilon(X_{P(0)}, L_{P(0)}; 1_{P(0)})\} \\ &\geq \min\{a_n + 1, \min_{1 \leq i \leq n-1} \frac{a_i + \dots + a_n + 1}{a_{i+1} + \dots + a_n + 1}\} \\ &= \min_{1 \leq i \leq n} \frac{a_i + \dots + a_n + 1}{a_{i+1} + \dots + a_n + 1}. \end{aligned}$$

We will use this lower bound in Section 5.

3.3. At a point in any orbit. Next, we consider the Seshadri constant on a toric variety at a point not necessarily contained in the maximal orbit.

Definition 3.13. Let P be an integral polytope of dimension n in $M_{\mathbb{R}}$, and v a vertex of P . We define

$$s(P; v) = \min\{|\tau|_{M_{\tau}} \mid v \prec \tau \prec P, \dim \tau = 1\} \in \mathbb{N} \setminus 0,$$

where $M_{\tau} = \mathbb{R}(\tau - \tau) \cap M$ and we consider τ as a subset in $(M_{\tau})_{\mathbb{R}} = \mathbb{R}(\tau - \tau)$ by a parallel translation. If $M = \{0\}$, we set $s(P; v) = +\infty$ for $P = v = \{0\}$.

Let σ be a face of P . Let $\pi : M_{\mathbb{R}} \rightarrow M_{\mathbb{R}}/\mathbb{R}(\sigma - \sigma)$ be the natural projection and set $M' = \pi(M)$, $P' = \pi(P)$, and $v' = \pi(v)$. Note that P' is an integral polytope in

$M'_\mathbb{R} = M_\mathbb{R}/\mathbb{R}(\sigma - \sigma)$ and v' is a vertex of P' . Then $s_1(P; \sigma), s_2(P; \sigma) \in \mathbb{R}_{>0}$ are defined to be

$$s_1(P; \sigma) = \min\{s_1^{M_\sigma}(\sigma), s(P'; v')\}, \quad s_2(P; \sigma) = \min\{s_2^{M_\sigma}(\sigma), s(P'; v')\},$$

where $M_\sigma = \mathbb{R}(\sigma - \sigma) \cap M$ and we regard σ as an integral polytope in $\mathbb{R}(\sigma - \sigma) = (M_\sigma)_\mathbb{R}$.

Note that $s_1(P; P) = s_1(P), s_2(P; P) = s_2(P)$, and $s_1(P; v) = s_2(P; v) = s(P; v)$ hold for any vertex v .

Proposition 3.14. *Let σ be a face of an n -dimensional integral polytope P in $M_\mathbb{R}$. Set $\pi : M_\mathbb{R} \rightarrow M_\mathbb{R}/\mathbb{R}(\sigma - \sigma), P' = \pi(P), v' = \pi(\sigma)$ as in Definition 3.13. Then,*

$$\varepsilon(X_P, L_P; p) = \min\{\varepsilon(X_\sigma, L_\sigma; 1_\sigma), s(P'; v')\}$$

holds for any $p \in O_\sigma$.

Proof. We use notations in Definition 3.13. We may assume $0 \in \sigma$, thus $v' = 0$ in $M' = \pi(M)$.

Firstly, we show $\varepsilon(X_P, L_P; p) \leq \min\{\varepsilon(X_\sigma, L_\sigma; p), s(P'; v')\}$. Note that $\varepsilon(X_\sigma, L_\sigma; p) = \varepsilon(X_\sigma, L_\sigma; 1_\sigma)$ by the torus action. Since $L_P|_{X_\sigma} = L_\sigma$, the inequality

$$\varepsilon(X_P, L_P; p) \leq \varepsilon(X_\sigma, L_\sigma; p) \cdots (*)$$

is clear. By the definition of π , there is a natural 1 to 1 correspondence between $\Xi := \{\tau \mid \sigma \prec \tau \prec P, \dim \tau = \dim \sigma + 1\}$ and $\Xi' := \{\tau' \mid v' \prec \tau' \prec P', \dim \tau' = 1\}$ by corresponding $\tau \in \Xi$ to $\pi(\tau) \in \Xi'$. Fix $\tau' \in \Xi'$ and let $\tau \in \Xi$ be the corresponding face of P . Then by Proposition 3.10, $\varepsilon(X_\tau, L_\tau; q) \leq s_2(\tau) \leq |\tau'|$ holds for $q \in O_\tau$. Since $\text{codim}(X_\sigma, X_\tau) = 1$ and X_τ is normal, X_τ is smooth at p . Therefore by the lower semicontinuity of Seshadri constants (see [La2, Example 5.1.11]), it holds that $\varepsilon(X_P, L_P; p) \leq \varepsilon(X_\tau, L_\tau; p) \leq \varepsilon(X_\tau, L_\tau; q) \leq |\tau'|$. Hence by definition of $s(P'; v')$,

$$\varepsilon(X_P, L_P; p) \leq \min_{\tau' \in \Xi'} |\tau'| = s(P'; v') \cdots (**)$$

From $(*)$ and $(**)$, we have $\varepsilon(X_P, L_P; p) \leq \min\{\varepsilon(X_\sigma, L_\sigma; p), s(P'; v')\}$.

Next we show the opposite inequality. Let C be a curve on X_P containing p . It is enough to show

$$C.L_P \geq \text{mult}_p(C) \cdot \min\{\varepsilon(X_\sigma, L_\sigma; p), s(P'; v')\} \cdots (***)$$

Case 1. $C \subset X_\sigma$.

In this case, $C.L_P \geq \text{mult}_p(C) \cdot \varepsilon(X_\sigma, L_\sigma; p)$ is clear by the definition of Seshadri constants, thus $(***)$ holds.

Case 2. $C \not\subset X_\sigma$.

We use the following claim:

Claim 3.15. *In this case, there exist $\tau' \in \Xi'$ and an effective divisor $D \in |L_P \otimes \mathfrak{m}_p^{|\tau'|}|$ on X_P such that $C \not\subset \text{Supp } D$.*

If there exists such a divisor D , it holds that

$$\begin{aligned} C.L_P = C.D &\geq \text{mult}_p(C) \cdot |\tau'| \\ &\geq \text{mult}_p(C) \cdot s(P'; v') \\ &\geq \text{mult}_p(C) \cdot \min\{\varepsilon(X_\sigma, L_\sigma; p), s(P'; v')\}. \end{aligned}$$

Thus the proof is completed by showing this claim.

Proof of Claim 3.15. For $\tau' \in \Xi'$, let $v'_{\tau'}$ be the vertex of τ' different from v' . If $\tau \in \Xi$ corresponds to τ' , there exists a vertex v_τ of τ such that $\pi(v_\tau) = v'_{\tau'}$. Many vertices of τ may satisfy this condition, but we choose one of them. Let $x^{v_\tau} \in H^0(X_P, L_P)$ be the section corresponding to v_τ , and $D_{v_\tau} \in |L_P|$ the corresponding effective divisor on X_P . Since $\text{Supp } D_{v_\tau} = \bigcup_{v_\tau \notin \rho \prec P} X_\rho$, we have

$$\bigcap_{\tau \in \Xi} \text{Supp } D_{v_\tau} = \bigcup_{v_\tau \notin \rho \text{ for } \forall \tau \in \Xi} X_\rho.$$

By the choices of v_τ , σ does not contain any v_τ . If $\rho \succ \sigma$ and $\rho \neq \sigma$, then ρ contains some $\tau \in \Xi$, hence $v_\tau \in \tau \subset \rho$. Consequently, it holds that

$$X_\sigma \subset \bigcap_{\tau \in \Xi} \text{Supp } D_{v_\tau} \subset X_\sigma \cup \bigcup_{\sigma \not\prec \rho \prec P} X_\rho.$$

Since $\bigcup_{\sigma \not\prec \rho \prec P} X_\rho$ is a closed set not containing p , $\bigcap_{\tau \in \Xi} \text{Supp } D_{v_\tau}$ coincides with X_σ around p . Now C contains p and is not contained in X_σ by assumption, thus C is not contained in $\bigcap_{\tau \in \Xi} \text{Supp } D_{v_\tau}$. Hence we can choose $\tau_0 \in \Xi$ such that $\text{Supp } D_{v_{\tau_0}}$ does not contain C . Let $\tau'_0 \in \Xi'$ be the corresponding face, and set $e' = |\tau'_0|^{-1} v'_{\tau'_0} \in M'$. (Note that we assume $v' = 0$, and τ'_0 is the convex hull of v' and $v'_{\tau'_0}$.) Then e' is the generator of $\mathbb{R}(\tau'_0 - \tau'_0) \cap M' = \mathbb{R}\tau'_0 \cap M' \cong \mathbb{Z}$ contained in τ'_0 . Fix $e \in M \cap \pi^{-1}(e')$. Since $v'_{\tau'_0} = |\tau'_0|e'$, $u := v_{\tau_0} - |\tau'_0|e$ is contained in $\pi^{-1}(0) \cap M = \mathbb{R}(\sigma - \sigma) \cap M$. This means $x^{v_{\tau_0}} = x^u \cdot (x^e)^{|\tau'_0|}$ is contained in $H^0(X_P, L_P \otimes \mathfrak{m}_p^{|\tau'_0|})$, hence this τ'_0 and $D_{v_{\tau_0}}$ satisfies the condition in the claim. \square

Remark 3.16. For a vertex v of P , we have $\varepsilon(X_P, L_P; p) = s(P; v)$ for the torus invariant point $p = O_v$ by Proposition 3.14. When X_P is smooth, this is Corollary 4.2.2 in [BDH+].

The invariant $s(P'; v')$ in Proposition 3.14 is easily computed. Thus, it is enough to see $\varepsilon(X_\sigma, L_\sigma; 1_\sigma)$ to compute $\varepsilon(X_P, L_P; p)$ for $p \in O_\sigma$. But we can use Proposition 3.10 to estimate $\varepsilon(X_\sigma, L_\sigma; 1_\sigma)$. Therefore we obtain the following theorem:

Theorem 3.17 (=Theorem 1.3). *Let $P \subset M_{\mathbb{R}}$ be an integral polytope, σ a face of P , and $p \in O_\sigma$. Then, it holds that*

$$s_1(P; \sigma) \leq \varepsilon(X_P, L_P; p) \leq s_2(P; \sigma).$$

Proof. This is easily shown from Propositions 3.10, 3.14, and the definitions of $s_1(P; \sigma)$ and $s_2(P; \sigma)$. \square

Remark 3.18. Since $s_1(\sigma) = s_2(\sigma)$ for $\sigma \subset (M_\sigma)_{\mathbb{R}}$ when $\text{rank } M_\sigma = 0$ or 1 , $s_1(P; \sigma) = \varepsilon(X_P, L_P; p) = s_2(P; \sigma)$ holds if $\dim \sigma = 0$ or 1 .

3.4. At a point in any orbit, Examples.

Example 3.19. Unless otherwise stated, π, P' and v' are as in Proposition 3.14.

(1) Let P_n be as in Example 3.12 (1). We apply Proposition 3.14 to P_n and any face $\sigma \prec P_n$ of codimension r . Then the image of P by π is $P' = P_r$, thus $s(P', v') = 1$. Since σ is identified with P_{n-r} by some integral affine translation, $\varepsilon(X_\sigma, L_\sigma; 1_\sigma) = 1$ holds by Example 3.12 (1). Hence we have $\varepsilon(X_{P_n}, L_{P_n}; p) = 1$ for any $\sigma \prec P$ and any $p \in O_\sigma$ by Proposition 3.14.

- (2) Let P be as in Example 3.12 (2). Then we have $\varepsilon(X_P, L_P; p) = a$ for any $p \in X_P$ by Proposition 3.14.
- (3) Let P be as in Example 3.12 (3). Then for any 1-dimensional face σ of P , $s(P', v') = |P'| = 3$ and $\varepsilon(X_\sigma, L_\sigma; 1_\sigma) = 1$. Thus $\varepsilon(X_P, L_P; p) = \min\{1, 3\} = 1$ for $p \in O_\sigma$. For any vertex v of P , $\varepsilon(X_P, L_P; p) = s(P; v) = 1$ by Remark 3.16. Thus we have $\varepsilon(X_P, L_P; p) = 1$ for $p \in X_P \setminus O_P$.
- (4) For an integral polytope $P \subset \mathbb{R}^2$ such that X_P is a Del pezzo surface and $L_P = -K_{X_P}$, we can easily compute $\varepsilon(X_P, L_P; p)$ for any p by Propositions 3.7 and 3.14. As a consequence, we know $\varepsilon(X_P, L_P; p) \in \{1, 2, 3\}$ for such P and any $p \in X_P$.
- (5) As (4), we can easily compute $\varepsilon(X_P, L_P; p)$ if X_P is a smooth toric Fano 3-fold and $L_P = -K_{X_P}$. As a consequence, we know $\varepsilon(X_P, L_P; p) \in \{1, 2, 3, 4\}$ for such P and any $p \in X_P$.
- (6) Let P be as in Example 3.12 (7), and $\sigma \prec P$ a 1-dimensional face. Then it is easy to see $s_1(P; \sigma) = s_2(P; \sigma) = \min\{|\sigma|, abc/|\sigma|\}$. Thus we have

$$\varepsilon(\mathbb{P}(a, b, c), \mathcal{O}(1); p) = (abc)^{-1} \min\{|\sigma|, abc/|\sigma|\} = \min\{|\sigma|/abc, 1/|\sigma|\}$$

for $p \in O_\sigma$. For example, $\varepsilon(\mathbb{P}(a, b, c), \mathcal{O}(1); p) = \min\{b/abc, 1/b\} = 1/ac$ for $p \in O_\sigma$ if σ is the convex hull of $(0, 0)$ and (ab, qb) . Note that $|\sigma|$ is not the Euclidean length of σ in \mathbb{R}^2 , i.e., $|\sigma|$ is not $b\sqrt{a^2 + q^2}$ but b .

When $\sigma \prec P$ is a vertex, we can easily compute $s(P; \sigma)$. For example,

$$\varepsilon(\mathbb{P}(a, b, c), \mathcal{O}(1); p) = s(P; \sigma) = \min\{1/bc, 1/ac\}$$

holds if $\sigma = (0, 0)$ and $p = O_\sigma$.

4. SESHADRI CONSTANTS AND TORIC DEGENERATIONS

In the above section, we study the Seshadri constants on toric varieties. In this section, we investigate non-toric cases by using toric degenerations.

Definition 4.1. Let L be a nef \mathbb{R} -divisor on a projective variety X and $\overline{m} = (m_1, \dots, m_r) \in \mathbb{R}_{>0}^r$ for $r > 0$. We say $L(\overline{m})$ or $L(m_1, \dots, m_r)$ is nef (resp. ample) if so is

$$\mu^*L - \sum_{i=1}^r m_i E_i,$$

where p_1, \dots, p_r are very general r points on X , $\mu : \tilde{X} \rightarrow X$ is the blowing up at p_1, \dots, p_r and E_i is the exceptional divisor over p_i . In other words, $L(\overline{m})$ is nef if and only if $\varepsilon(X, L; \overline{m}) \geq 1$. We sometimes denote $\mu^*L - \sum_{i=1}^r m_i E_i$ by $L(\overline{m})$ for very general p_i .

Remark 4.2. To show the nefness of $L(\overline{m})$, it is enough to show $\mu^*L - \sum_{i=1}^r m_i E_i$ is nef for one choice of p_1, \dots, p_r . This follows from the openness of the ampleness condition as in [Bi, Lemma.6.1.A].

By using degenerations, we can show the nefness (resp. ampleness) of a divisor from the nefness (resp. ampleness) of other divisors. The following theorem is a straightforward generalization of Theorem 2.A in [Bi]:

Theorem 4.3. Let $f : \mathcal{X} \rightarrow T$ be a flat projective morphism over a smooth variety T with reduced and irreducible general fibers, and \mathcal{L} an f -nef (resp. f -ample) divisor on \mathcal{X} . Let $X_t = f^{-1}(t)$ be the scheme theoretic fiber of f , $L_t = \mathcal{L}|_{X_t}$ for $t \in T$. Assume that

Y_i ($1 \leq i \leq r$) are irreducible components of the central fiber X_0 ($0 \in T$) with the reduced structures (other components may exist). We assume the following:

- (i) X_0 is reduced at the generic point of Y_i for any i ,
- (ii) There exist $k_i \in \mathbb{N}$ and $\overline{m}^{(i)} = (m_1^{(i)}, \dots, m_{k_i}^{(i)}) \in \mathbb{R}_{>0}^{k_i}$ for $1 \leq i \leq r$ such that $\mathcal{L}|_{Y_i}(\overline{m}^{(i)})$ is nef (resp. ample) for any i .

Then $L_t(\overline{m}^{(1)}, \dots, \overline{m}^{(r)})$ is nef (resp. ample) for very general $t \in T$.

Proof. Fix very general points $p_1^{(i)}, \dots, p_{k_i}^{(i)}$ in Y_i for $i = 1, \dots, r$.

Firstly, we assume that there exist sections of f , $\{\sigma_j^{(i)}\}$ for $1 \leq i \leq r$, $1 \leq j \leq k_i$ satisfying $\sigma_j^{(i)}(0) = p_j^{(i)}$. By shrinking T if necessary, we may assume $\sigma_j^{(i)}(T) \cap \sigma_{j'}^{(i')}(T) = \emptyset$ for $(i, j) \neq (i', j')$. Let $\mu : \mathcal{X}' \rightarrow \mathcal{X}$ be the blowing up along $\bigcup_{i,j} \sigma_j^{(i)}(T)$, $\mathcal{E}_j^{(i)}$ the exceptional divisor over $\sigma_j^{(i)}(T)$, and set $\mathcal{L}' = \mu^* \mathcal{L} - \sum_{i,j} m_j^{(i)} \mathcal{E}_j^{(i)}$. Then for very general t ,

$$\mu_t : (f \circ \mu)^{-1}(t) \rightarrow f^{-1}(t) = X_t$$

is the blowing up along $\sum_{i=1}^r k_i$ smooth points $\{\sigma_j^{(i)}(t)\}$, and it holds that

$$\mathcal{L}'|_{(f \circ \mu)^{-1}(t)} = \mu_t^* L_t - \sum_{i,j} m_j^{(i)} E_{j,t}^{(i)},$$

where $E_{j,t}^{(i)}$ is the exceptional divisor over $\sigma_j^{(i)}(t)$. By the assumption ii) and the choice of $p_j^{(i)}$, the restriction of \mathcal{L}' on the fiber of $f \circ \mu : \mathcal{X}' \rightarrow T$ over 0 is nef (resp. ample). Hence $\mathcal{L}'|_{(f \circ \mu)^{-1}(t)} = \mu_t^* L_t - \sum_{i,j} m_j^{(i)} E_{j,t}^{(i)}$ is also nef (resp. ample) for very general $t \in T$. Thus $L_t(\overline{m}^{(1)}, \dots, \overline{m}^{(r)})$ is nef (resp. ample).

In general there may not exist such sections, but we can make sections by a base change as follows.

From the assumption i) and by cutting by sufficiently ample divisors on \mathcal{X} , there exists a subvariety $U \subset \mathcal{X}$ such that U contains all $p_j^{(i)}$ and the restriction $f|_U : U \rightarrow T$ is étale at $p_j^{(i)}$ for any i, j . Set $U^{(k)} = \underbrace{U \times_T \dots \times_T U}_k$ for $k \in \mathbb{N}$. Then the natural morphism

$\alpha : U^{(\sum k_i)} = U^{(k_1)} \times_T \dots \times_T U^{(k_r)} \rightarrow T$ is étale at $\tilde{p} = (p^{(i)})_i \in U^{(k_1)} \times_T \dots \times_T U^{(k_r)}$, where $p^{(i)} = (p_j^{(i)})_j \in U^{(k_i)}$. Thus there is an open neighborhood $V \subset U^{(k_1)} \times_T \dots \times_T U^{(k_r)}$ of \tilde{p} such that $\alpha|_V : V \rightarrow T$ is étale. Then by base change we have a diagram

$$\begin{array}{ccc} \mathcal{X} \times_T V & \xrightarrow{\beta} & \mathcal{X} \\ \downarrow g & \circlearrowleft & \downarrow f \\ V & \xrightarrow{\alpha|_V} & T. \end{array}$$

Since $\alpha|_V$ is étale, $g^{-1}(v) \cong f^{-1}(\alpha(v)) = X_{\alpha(v)}$ for $v \in V$, and g and $\beta^* \mathcal{L}$ satisfy the conditions i) and ii) for the central fiber $g^{-1}(\tilde{p}) = X_0$ and Y_i . (Note $\alpha(\tilde{p}) = 0 \in T$.) The morphism

$$V \hookrightarrow U^{(k_1)} \times_T \dots \times_T U^{(k_r)} \xrightarrow{\pi_i} U^{(k_i)} \xrightarrow{\varpi_j} U \hookrightarrow \mathcal{X}$$

induces a section $\sigma_j^{(i)}$ of g , where π_i and ϖ_j are the i -th and j -th projections respectively. Since $\sigma_j^{(i)}(\tilde{p}) = p_j^{(i)} \in X_0 = g^{-1}(\tilde{p})$ by the definition of $\sigma_j^{(i)}$, we can use the first part of this proof. Thus, $(\beta^* \mathcal{L})|_{g^{-1}(v)}(\overline{m}^{(1)}, \dots, \overline{m}^{(r)})$ is nef (resp. ample) for very

general $v \in V$. If we identify $X_{\alpha(v)}$ with $g^{-1}(v)$, $L_{\alpha(v)}(\overline{m}^{(1)}, \dots, \overline{m}^{(r)})$ is identified with $(\beta^* \mathcal{L})|_{g^{-1}(v)}(\overline{m}^{(1)}, \dots, \overline{m}^{(r)})$. Since $\alpha|_V$ is étale, particularly generically surjective, the nefness (reps. ampleness) of $L_t(\overline{m}^{(1)}, \dots, \overline{m}^{(r)})$ follows for very general $t \in T$. \square

Corollary 4.4. *Let $f : \mathcal{X} \rightarrow T, \mathcal{L}, X_0$ and Y_i be as in Theorem 4.3 satisfying condition (i). Moreover assume that there exists an integral polytope P_i such that the normalization of $(Y_i, \mathcal{L}|_{Y_i})$ is isomorphic to (X_{P_i}, L_{P_i}) as a polarized variety for each $1 \leq i \leq r$. Then $L_t(\varepsilon_1, \dots, \varepsilon_r)$ is nef for very general $t \in T$, where $\varepsilon_i = \varepsilon(X_{P_i}, L_{P_i}; 1_{P_i})$.*

In particular, $L_t(s_1(P_1), \dots, s_1(P_r))$ is nef for very general $t \in T$.

Proof. Since the normalization is isomorphic over a non-empty open set in Y_i , it holds that $\varepsilon(X_{P_i}, L_{P_i}; 1) = \varepsilon(Y_i, \mathcal{L}|_{Y_i}; 1)$. Applying Theorem 4.3 to $k_i = 1, m_1^{(i)} = \varepsilon_i$, the nefness of $L_t(\varepsilon_1, \dots, \varepsilon_r)$ follows for very general $t \in T$. The last statement is clear from $\varepsilon_i \geq s_1(P_i)$. \square

5. EXAMPLES IN NON-TORIC CASES

Theorem 4.3 and Corollary 4.4 tell us a strategy for obtaining lower bounds of (multi-point) Seshadri constants at very general points:

Finding degenerations to (unions of) polarized varieties whose Seshadri constants are more computable, such as toric varieties.

Toric degenerations are studied very well, thus we know many such degenerations. Furthermore the assumption that the normalizations are toric in Corollary 4.4 is weaker than usual toric degenerations, which assume the irreducible components are normal toric themselves. Therefore we can find more such degenerations. Of course, we do not know when such degenerations exist in general. The obtained lower bounds may not be good even if such degenerations exist. But if we can find good degenerations, we sometimes get good lower bounds as we will see in the rest of this paper.

In this section, we estimate Seshadri constants on some non-toric varieties by using Theorem 4.3 and Corollary 4.4.

5.1. Hypersurfaces and complete intersections in projective spaces. In this subsection, we study Seshadri constants on hypersurfaces or complete intersections in projective spaces. For positive integers d_1, \dots, d_k and n , we denote by X_{d_1, \dots, d_k}^n a very general complete intersection of hypersurfaces of degrees d_1, \dots, d_k in \mathbb{P}^{n+k} .

Firstly, we estimate $\varepsilon(X, \mathcal{O}(1); 1)$ for a very general complete intersection X :

Proposition 5.1. *Let d_1, \dots, d_k and n be positive integers. Suppose that there exist a positive integer c and natural numbers l_1, \dots, l_k such that $\sum_{j=1}^k l_j = n$ and $d_j \geq c^{l_j}$ hold for any $1 \leq j \leq k$. Then $\varepsilon(X_{d_1, \dots, d_k}^n, \mathcal{O}(1); 1) \geq c$ holds.*

In particular, $\varepsilon(X_d^n, \mathcal{O}(1); 1) \geq \lfloor \sqrt[n]{d} \rfloor$ holds for any $d \in \mathbb{N} \setminus 0$.

Proof. We prove this proposition by 3 steps.

Step 1. Firstly, we find a not necessarily normal toric variety which is a complete intersection of hypersurfaces of degrees d_1, \dots, d_k in \mathbb{P}^{n+k} . Let $d_j^{(i)}$ be natural numbers for $1 \leq i \leq n, 1 \leq j \leq k$ such that $1 + \sum_{i=1}^n d_j^{(i)} = d_j$ holds for any j . We consider the following

homogeneous polynomials

$$\begin{aligned} T_0^{d_1} &= T_1^{d_1^{(1)}} T_2^{d_1^{(2)}} \cdots T_n^{d_1^{(n)}} T_{n+1} \\ T_{n+1}^{d_2} &= T_1^{d_2^{(1)}} T_2^{d_2^{(2)}} \cdots T_n^{d_2^{(n)}} T_{n+2} \\ &\vdots \\ T_{n+k-1}^{d_k} &= T_1^{d_k^{(1)}} T_2^{d_k^{(2)}} \cdots T_n^{d_k^{(n)}} T_{n+k}, \end{aligned}$$

where T_0, \dots, T_{n+k} are the homogeneous coordinates of \mathbb{P}^{n+k} . It is not hard to see that the intersection X of the hypersurfaces defined by these polynomials is reduced, irreducible, and n -dimensional, i.e., a complete intersection variety in \mathbb{P}^{n+k} .

Set P be the image of $\text{conv}(0, e_1, \dots, e_{n+k}) \subset \mathbb{R}^{n+k}$ by the lattice projection

$$\pi : \mathbb{R}^{n+k} = (\mathbb{Z}^{n+k})_{\mathbb{R}} \rightarrow (\mathbb{Z}^{n+k}/M)_{\mathbb{R}},$$

where M is the subgroup of \mathbb{Z}^{n+k} spanned by $\sum_{i=1}^n d_1^{(i)} e_i + e_{n+1}$ and $\sum_{i=1}^n d_j^{(i)} e_i - d_j e_{n+j-1} + e_{n+j}$ for $2 \leq j \leq k$. Note that $\sum_{i=1}^n d_1^{(i)} e_i + e_{n+1}$ comes from $T_0^{d_1} = T_1^{d_1^{(1)}} T_2^{d_1^{(2)}} \cdots T_n^{d_1^{(n)}} T_{n+1}$ for example. By Lemma 3.4, $(X, \mathcal{O}(1))$ is a not necessarily normal toric variety whose normalization is (X_P, L_P) . Since X_{d_1, \dots, d_k}^n degenerates to X , we have $\varepsilon(X_{d_1, \dots, d_k}^n, \mathcal{O}(1); 1) \geq \varepsilon(X_P, L_P; 1)$ by Corollary 4.4. Thus it suffices to show $\varepsilon(X_P, L_P; 1) \geq c$ for a suitable choice of $d_j^{(i)}$.

Step 2. Secondly, we estimate $\varepsilon(X_P, L_P; 1)$ by $d_j^{(i)}$. We denote $\pi(e_l)$ by $[e_l]$ for $1 \leq l \leq n+k$. Since the coefficient of e_{n+1} in $\sum_{i=1}^n d_1^{(i)} e_i + e_{n+1}$ and that of e_{n+j} in $\sum_{i=1}^n d_j^{(i)} e_i - d_j e_{n+j-1} + e_{n+j}$ are 1 for $2 \leq j \leq k$, we can take $[e_1], \dots, [e_n]$ as a basis of \mathbb{Z}^{n+k}/M . It is easy to see that P is the convex hull of $[e_1], \dots, [e_n]$, and $[e_{n+k}]$. Since $[e_{n+1}] = -\sum_{i=1}^n d_1^{(i)} [e_i]$ and $[e_{n+j}] = -\sum_{i=1}^n d_j^{(i)} [e_i] + d_j [e_{n+j-1}]$, we can show $[e_{n+k}] = -\sum_{i=1}^n a^{(i)} e_i$ for

$$\begin{aligned} a^{(i)} &= \sum_{j=1}^k d_j^{(i)} d_{j+1} \cdots d_k \\ &= d_1^{(i)} d_2 \cdots d_k + \cdots + d_{k-1}^{(i)} d_k + d_k^{(i)}. \end{aligned}$$

By Example 3.12 (8), we have

$$\varepsilon(X_P, L_P; 1) \geq \min_{1 \leq i \leq n} b^{(i)} / b^{(i+1)},$$

where $b^{(i)} = a^{(i)} + a^{(i+1)} + \cdots + a^{(n)} + 1$ for $1 \leq i \leq n$ and $b^{(n+1)} = 1$.

Step 3. Note that $X_{d_1, \dots, d_k}^n = \bigcap_j X_{d_j}^{n+k-1}$ degenerates to $\bigcap_j (X_{c^{l_j}}^{n+k-1} \cup X_{d_j - c^{l_j}}^{n+k-1})$. Since $X_{c^{l_j}}^{n+k-1}$ and $X_{d_j - c^{l_j}}^{n+k-1}$ are very general, $X_{c^{l_1}, \dots, c^{l_k}}^n = \bigcap_j X_{c^{l_j}}^{n+k-1}$ is an irreducible component of $\bigcap_j (X_{c^{l_j}}^{n+k-1} \cup X_{d_j - c^{l_j}}^{n+k-1})$. By applying Theorem 4.3 to this degeneration, we have

$$\varepsilon(X_{d_1, \dots, d_k}^n, \mathcal{O}(1); 1) \geq \varepsilon(X_{c^{l_1}, \dots, c^{l_k}}^n, \mathcal{O}(1); 1).$$

Thus it is enough to show this proposition for $d_j = c^{l_j}$.

Let us define $d_j^{(i)}$ for $d_j = c^{l_j}$ such that $b^{(i)}/b^{(i+1)} = c$ for any $1 \leq i \leq n$. Set $d_j^{(i)}$ as follows:

$$d_j^{(i)} = \begin{cases} (c-1)c^{h_j-i} & \text{if } h_{j-1} < i \leq h_j \\ 0 & \text{otherwise,} \end{cases}$$

where $h_j = l_1 + \dots + l_j$ for $1 \leq j \leq k$ and $h_0 = 0$. Note $h_k = \sum_j l_j = n$ and $\sum_i d_j^{(i)} = c^{l_j} - 1 = d_j - 1$. For each i , we define j_i to be the unique j satisfying $h_{j-1} < i \leq h_j$. Then we have

$$a^{(i)} = d_{j_i+1} \cdots d_k (c-1)c^{h_{j_i}-i} = (c-1)c^{n-i}.$$

Since $b^{(i)} = a^{(i)} + b^{(i+1)}$, we have $b^{(i)} = c^{n+1-i}$. Thus $b^{(i)}/b^{(i+1)} = c^{n+1-i}/c^{n-i} = c$, which proves this proposition. \square

Example 5.2. If we choose $d_j^{(i)}$ carefully, we may obtain a better estimation than that of Proposition 5.1. We use notations as in the proof of Proposition 5.1.

(1) Let $2 \leq d_1 \leq \dots \leq d_k$ be positive integers such that $\sum_j d_j \leq n+k$. Then X_{d_1, \dots, d_k}^n is a Fano n -fold such that $-K_{X_{d_1, \dots, d_k}^n} = \mathcal{O}(n+k+1 - \sum_j d_j)$. If $\sum_j d_j < n+k$, it is known that X_{d_1, \dots, d_k}^n is covered by lines (cf. [Deb, Proposition 2.13]). Hence we have $\varepsilon(X_{d_1, \dots, d_k}^n, \mathcal{O}(1); 1) = 1$.

Now assume $\sum_j d_j = n+k$. Then X_{d_1, \dots, d_k}^n is a Fano n -fold such that $-K_{X_{d_1, \dots, d_k}^n} = \mathcal{O}(1)$. We can show $\varepsilon(X_{d_1, \dots, d_k}^n, \mathcal{O}(1); 1) = d_k/(d_k - 1)$ as follows:

We define $d_j^{(i)}$ by

$$d_j^{(i)} = \begin{cases} 1 & \text{if } h'_{j-1} < i \leq h'_j \\ 0 & \text{otherwise,} \end{cases}$$

where $h'_j = (d_1 - 1) + \dots + (d_j - 1)$. Note $h'_k = \sum_{j=1}^k (d_j - 1) = \sum_j d_j - k = n$. Then we have $a^{(i)} = d_{j+1} \cdots d_k$ and $b^{(i)} = d_{j+1} \cdots d_k (h'_j + 2 - i)$ for $h'_{j-1} < i \leq h'_j$. By Steps 1 and 2 in the proof of Theorem 5.1, it holds that

$$\varepsilon(X_{d_1, \dots, d_k}^n, \mathcal{O}(1); 1) \geq \min_{1 \leq i \leq n} \frac{b^{(i)}}{b^{(i+1)}} = \min_{1 \leq j \leq k} \frac{d_j}{d_j - 1} = \frac{d_k}{d_k - 1}.$$

Next, we show $\varepsilon(X_{d_1, \dots, d_k}^n, \mathcal{O}(1); 1) \leq d_k/(d_k - 1)$ by finding a curve $C \subset X_{d_1, \dots, d_k}^n$ such that $C \cdot \mathcal{O}(1) / \text{mult}_p(C) = d_k/(d_k - 1)$ for any very general point $p \in X_{d_1, \dots, d_k}^n$.

Let F_1, \dots, F_k be homogeneous polynomials in $\mathbb{C}[T_0, \dots, T_{n+k}]$ of degrees d_1, \dots, d_k respectively such that $X_{d_1, \dots, d_k}^n = (F_1 = \dots = F_k = 0)$. We may assume $p = [1 : 0 : \dots : 0] \in \mathbb{P}^{n+k}$. Then there exist homogeneous polynomials $F_j^i \in \mathbb{C}[T_1, \dots, T_{n+k}]$ such that $\deg F_j^i = i$ and $F_j = \sum_{i=1}^{d_j} T_0^{d_j-i} F_j^i$. Let D_j and $D_j^i \subset \mathbb{P}^{n+k}$ be the hypersurfaces defined by F_j and F_j^i respectively. Then

$$\bigcap_{i=1}^{d_j} D_j^i = (F_j^1 = \dots = F_j^{d_j} = 0) \subset D_j$$

for $1 \leq j \leq k-1$, and

$$\bigcap_{i=1}^{d_k-2} D_k^i \cap (T_0 F_k^{d_k-1} + F_k^{d_k} = 0) = (F_k^1 = \dots = F_k^{d_k-2} = T_0 F_k^{d_k-1} + F_k^{d_k} = 0) \subset D_k.$$

Note that all F_j^i are general since X_{d_1, \dots, d_k}^n and p are general. Hence

$$C := \bigcap_{j=1}^{k-1} \bigcap_{i=1}^{d_j} D_j^i \cap \left(\bigcap_{i=1}^{d_k-2} D_k^i \cap (T_0 F_k^{d_k-1} + F_k^{d_k} = 0) \right)$$

is a complete intersection curve in \mathbb{P}^{n+k} , and $p \in C \subset \bigcap_{j=1}^k D_j = X_{d_1, \dots, d_k}^n$. By definition,

$$\deg C = \prod_{j=1}^{k-1} d_j! \cdot (d_k - 2)! \cdot d_k$$

and

$$\text{mult}_p(C) = \prod_{j=1}^{k-1} d_j! \cdot (d_k - 1)!.$$

Thus we have $\varepsilon(X_{d_1, \dots, d_k}^n, \mathcal{O}(1); 1) = \varepsilon(X_{d_1, \dots, d_k}^n, \mathcal{O}(1); p) \leq \deg C / \text{mult}_p(C) = d_k / (d_k - 1)$. Therefore $\varepsilon(X_{d_1, \dots, d_k}^n, \mathcal{O}(1); 1) = d_k / (d_k - 1)$ holds.

For example, we have

$$\begin{aligned} \varepsilon(X_4^3, \mathcal{O}(1); 1) &= 4/3, \\ \varepsilon(X_{2,3}^3, \mathcal{O}(1); 1) &= 3/2, \\ \varepsilon(X_{2,2,2}^3, \mathcal{O}(1); 1) &= 2 \end{aligned}$$

when $n = 3$.

(2) When $k = 1$, we denote $d = d_1, d^{(i)} = d_1^{(i)}$ for simplicity. Then, $a^{(i)} = d^{(i)}$ for any i . Thus we have

$$\varepsilon(X_d^n, \mathcal{O}(1); 1) \geq \min_{1 \leq i \leq n} \frac{d^{(i)} + \dots + d^{(n)} + 1}{d^{(i+1)} + \dots + d^{(n)} + 1}.$$

In other words,

$$\varepsilon(X_d^n, \mathcal{O}(1); 1) \geq \min \left\{ \frac{c_n}{1}, \frac{c_{n-1}}{c_n}, \dots, \frac{c_2}{c_3}, \frac{c_1}{c_2} \right\}$$

holds for any increase sequence of positive integers $1 \leq c_n \leq c_{n-1} \leq \dots \leq c_1 = d$.

When $n = 2$, set $c_1 = d, c_2 = \lceil \sqrt{d} \rceil$. Then $\varepsilon(X_d^2, \mathcal{O}(1); 1) \geq \min\{\lceil \sqrt{d} \rceil, d/\lceil \sqrt{d} \rceil\} = d/\lceil \sqrt{d} \rceil$ holds. From this and Proposition 5.1, we have

$$\varepsilon(X_d^2, \mathcal{O}(1); 1) \geq \max\{\lfloor \sqrt{d} \rfloor, d/\lceil \sqrt{d} \rceil\}.$$

When $d \geq 4$, $\varepsilon(X_d^2, \mathcal{O}(1); 1) \geq \lfloor \sqrt{d} \rfloor$ follows from Proposition 1 in [St] as well since $\text{Pic } X = \mathbb{Z}\mathcal{O}_X(1)$. But $d/\lceil \sqrt{d} \rceil$ is a new estimation. For example, $\varepsilon(X_7^2, \mathcal{O}(1); 1) \geq 7/3$ holds.

Remark 5.3. "In particular" part of Proposition 5.1, i.e., $\varepsilon(X_d^n, \mathcal{O}(1); 1) \geq \lfloor \sqrt[n]{d} \rfloor$, can be shown easily without using estimations in toric cases as follows.

By the first statement of Step 3 in the proof of Proposition 5.1, we may assume $d = c^n$ for $c \in \mathbb{N} \setminus 0$. We consider the embedding $i : \mathbb{P}^n \hookrightarrow \mathbb{P}^N$ defined by $|\mathcal{O}_{\mathbb{P}^n}(c)|$. By a suitable linear projection $\pi : \mathbb{P}^N \dashrightarrow \mathbb{P}^{n+1}$, $\pi \circ i : \mathbb{P}^n \rightarrow \mathbb{P}^{n+1}$ is a finite birational morphism onto the image $Y := \pi \circ i(\mathbb{P}^n)$. Thus $\varepsilon(Y, \mathcal{O}(1); 1) = \varepsilon(\mathbb{P}^n, \mathcal{O}(c); 1) = c$ holds since $(\pi \circ i)^* \mathcal{O}_Y(1) = \mathcal{O}_{\mathbb{P}^n}(c)$. By definition Y is a hypersurface of degree d in \mathbb{P}^{n+1} , hence X_d^n degenerates to Y . Therefore we have $\varepsilon(X_d^n, \mathcal{O}(1); 1) \geq \varepsilon(Y, \mathcal{O}(1); 1) = c = \lfloor \sqrt[n]{d} \rfloor$ by the lower semicontinuity of Seshadri constants.

Similarly, we can show $\varepsilon(X_{c^n d}^n, \mathcal{O}(1); \overline{m}) \geq c \varepsilon(X_d^n, \mathcal{O}(1); \overline{m})$ for any $c, d \in \mathbb{N} \setminus 0$ and $\overline{m} \in \mathbb{R}_{>0}^r$ by considering $X_d^n \xrightarrow{|\mathcal{O}(c)|} \mathbb{P}^N \dashrightarrow \mathbb{P}^{n+1}$.

Next, we study multi-point cases. The following proposition looks like Theorem 2.A in [Bi] somehow:

Proposition 5.4. *Let d_1, \dots, d_k, a, b , and n be positive integers for $k \in \mathbb{N}$. We denote by L_a, L_b , and L_{a+b} the invertible sheaf $\mathcal{O}(1)$ on $X_{d_1, \dots, d_k, a}^n, X_{d_1, \dots, d_k, b}^n$, and $X_{d_1, \dots, d_k, a+b}^n$ respectively.*

If $L_a(\overline{m}_1)$ and $L_b(\overline{m}_2)$ are nef (resp. ample) for $\overline{m}_1 \in \mathbb{R}_{>0}^{r_1}$ and $\overline{m}_2 \in \mathbb{R}_{>0}^{r_2}$, then $L_{a+b}(\overline{m}_1, \overline{m}_2)$ is also nef (resp. ample).

Proof. A very general hypersurface $X_{a+b}^{n+k} \subset \mathbb{P}^{n+k+1}$ of degree $a+b$ degenerates to the union $X_a^{n+k} \cup X_b^{n+k}$ of hypersurfaces of degrees a and b . Thus $X_{d_1, \dots, d_k, a+b}^n = X_{d_1, \dots, d_k}^{n+1} \cap X_{a+b}^{n+k}$ degenerates to $X_{d_1, \dots, d_k}^{n+1} \cap (X_a^{n+k} \cup X_b^{n+k}) = X_{d_1, \dots, d_k, a}^n \cup X_{d_1, \dots, d_k, b}^n$. Applying Theorem 4.3 to this degeneration, the proposition follows. \square

As a corollary of Proposition 5.1 and Theorem 4.3 or Proposition 5.4, we obtain estimations of multi-point Seshadri constants on hypersurfaces in projective spaces:

Theorem 5.5 (=Theorem 1.8). *Let X_d^n be a very general hypersurface of degree d in \mathbb{P}^{n+1} . Then it holds that*

$$\lfloor \sqrt[n]{d/(m_1^n + \dots + m_r^n)} \rfloor \leq \varepsilon(X_d^n, \mathcal{O}(1); \overline{m}) \leq \sqrt[n]{d/(m_1^n + \dots + m_r^n)}$$

for any $\overline{m} = (m_1, \dots, m_r) \in (\mathbb{N} \setminus 0)^r$.

Proof. The second inequality is clear since $\mathcal{O}_{X_d^n}(1)^n = d$. Thus it remains to prove the first inequality. For simplicity, set $c = \lfloor \sqrt[n]{d/(m_1^n + \dots + m_r^n)} \rfloor$. Let d_1, \dots, d_r be natural numbers such that $d = d_1 + \dots + d_r$ and $d_i \geq (cm_i)^n$. We can choose such d_i because $d \geq \sum_i (cm_i)^n$. By Proposition 5.1, $\varepsilon(X_{d_i}^n, \mathcal{O}(1); 1) \geq cm_i$ holds. Note that cm_i is an integer. Since X_d^n degenerates to $\bigcup_{i=1}^r X_{d_i}^n$, we can apply Theorem 4.3 and we know $L_d(cm_1, \dots, cm_r)$ is nef, where L_d is the invertible sheaf $\mathcal{O}(1)$ on X_d^n . Hence $\varepsilon(X_d^n, \mathcal{O}(1); \overline{m}) \geq c$ holds. \square

5.2. Fano 3-folds with Picard number 1. In this subsection, we estimate Seshadri constants on a smooth Fano 3-fold X with Picard number 1, i.e., X is a smooth projective variety of dimension 3 such that $-K_X$ is ample and $\text{Pic } X \cong \mathbb{Z}$. The index of X is the positive integer r such that $-K_X = rH$, where $H \in \text{Pic } X$ is the ample generator.

Toric degenerations of Fano 3-folds are studied by many authors. Small toric degenerations of Fano 3-folds are treated by [Ga], and [CI] investigated complete intersection cases in (weighted) projective spaces and homogeneous spaces. In [ILP], Ilten, Lewis, and Przyjalkowski studied remaining cases of Fano 3-folds with Picard number 1. They showed that every smooth Fano 3-fold of Picard number 1 has a toric degeneration and gave an explicit description of the moment polytope of the central fiber. Most of the degenerations in [ILP] give good lower bounds of Seshadri constants.

Example 5.6. Let $X \subset \mathbb{P}(1, 1, 1, 1, 3)$ be a very general hypersurface of degree 6. By [ILP, First Main Theorem], $(X, \mathcal{O}(1))$ degenerates to (X_P, L_P) (as a \mathbb{Q} -polarized variety) for $P := \text{conv}(e_1, e_2, e_3, -1/3(e_1 + e_2 + e_3)) \subset \mathbb{R}^3$. It is easy to see $s_1(P) \geq 6/5$. Thus we have $\varepsilon(X, \mathcal{O}(1); 1) \geq 6/5$ by Corollary 4.4.

We can show $\varepsilon(X, \mathcal{O}(1); 1) \leq 6/5$ by similar arguments as Example 5.2 (1), but we give a little more geometrical proof here.

Fix a very general point $p \in X$. Define $p' \in X$ by $\{p, p'\} := \varphi^{-1}(\varphi(p))$, where $\varphi : X \rightarrow \mathbb{P}^3$ is the double cover defined by $|\mathcal{O}_X(1)|$. Since $\dim H^0(X, \mathcal{O}(3)) = 21$ and $\dim \mathcal{O}_X/\mathfrak{m}_p^4 = 20$, there exists $S \in |\mathcal{O}_X(3) \otimes \mathfrak{m}_p^4|$. Then $\text{mult}_p(S) = 4$ because X and p are very general. It is not hard to see that S does not contain p' . Let $\pi : \tilde{X} \rightarrow X$ be the blowing up at $\{p, p'\}$, and set E, E' be the exceptional divisors over p and p' respectively. Let $\tilde{S} \subset \tilde{X}$ be the strict transform of S , and set $\psi = \varphi|_{\tilde{S}} : \tilde{S} \rightarrow S$ and $F = E|_{\tilde{S}}$. Then $F^2 = -\text{mult}_p(S) = -4$. Since $\varphi^*\mathcal{O}_X(1) - E - E'$ is base point free, so is $(\varphi^*\mathcal{O}_X(1) - E - E')|_{\tilde{S}} = \psi^*\mathcal{O}_S(1) - F$. Let $f : \tilde{S} \rightarrow \mathbb{P}^2$ be the morphism defined by $\varphi^*\mathcal{O}_S(1) - F$. By $\varepsilon(S, \mathcal{O}(1); p) \geq \varepsilon(X, \mathcal{O}(1); p) > 1$, we know $\varphi^*\mathcal{O}_S(1) - F$ is ample. Thus f is a finite morphism. Since $f_*F \cdot \mathcal{O}_{\mathbb{P}^2}(1) = F \cdot f^*\mathcal{O}_{\mathbb{P}^2}(1) = F \cdot (\varphi^*\mathcal{O}_S(1) - F) = 4$, we have $f_*F \sim \mathcal{O}_{\mathbb{P}^2}(4)$. Thus $D := f_*f^*F - F$ is an effective divisor and $D \sim \psi^*\mathcal{O}_S(4) - 5F$. Hence $\psi^*\mathcal{O}_S(1) - 6/5F$ is not ample because $D \cdot (\psi^*\mathcal{O}_S(1) - 6/5F) = (\psi^*\mathcal{O}_S(4) - 5F) \cdot (\psi^*\mathcal{O}_S(1) - 6/5F) = 0$. Thus $\varepsilon(X, \mathcal{O}(1); 1) = \varepsilon(X, \mathcal{O}(1); p) \leq \varepsilon(S, \mathcal{O}(1); p) \leq 6/5$ holds and we have $\varepsilon(X, \mathcal{O}(1), 1) = 6/5$.

It is known that there are 17 families of smooth Fano 3-folds with Picard number 1. For each case, we can compute the Seshadri constant as follows:

Theorem 5.7 (=Theorem 1.10). *For each family of smooth Fano 3-folds with Picard number 1, $\varepsilon(X, -K_X; 1)$ is as in Table 1, where X is a very general member in the family.*

Proof. For No.1-4 in Table 1, $\varepsilon(X, -K_X; 1)$ is computed in Examples 5.2 (1) and 5.6. (In fact, degenerations in [ILP] for No.2-4 give same lower bounds as Examples 5.2 (1) though some of their degenerations are different from those of Examples 5.2 (1).)

For No.5-17, we can show the following:

$$\varepsilon(X, -K_X; 1) \geq \begin{cases} 2 & \text{for No.5-15} \\ 3 & \text{for No.16} \\ 4 & \text{for No.17.} \end{cases} \quad \dots (*)$$

Except No.11, these lower bounds are obtained by applying Corollary 4.4 to the degenerations in [ILP, First Main Theorem].

In No.11 case, the moment polytope of the central fiber of the degeneration in [ILP] is $P' = \text{conv}(e_3, 2e_1 - e_3, e_2 - e_3, -2/3e_1 - 2/3e_2 - e_3)$. By the 2nd projection, we have $s_1(P') \leq s_2(P') \leq 5/3$. Thus $s_1(P')$ is not so large. Instead of this degeneration, we consider the following degeneration, whose construction is essentially same as Proposition 5.1.

Let T_0, T_1, T_2, T_3, T_4 be weighted homogeneous coordinates on $\mathbb{P}(1, 1, 1, 2, 3)$ with $\deg T_0 = \deg T_1 = \deg T_2 = 1, \deg T_3 = 2, \deg T_4 = 3$. Then $X_0 := (T_4^2 = T_1^2 T_2^2 T_3) \subset \mathbb{P}(1, 1, 1, 2, 3)$ is a non-normal toric variety whose moment polytope is $P = \text{conv}(0, e_1, e_2, -e_1 - e_2 + e_3)$. A very general hypersurface X in $\mathbb{P}(1, 1, 1, 2, 3)$ of degree 6 degenerates to X_0 . Since $-K_X = \mathcal{O}_X(2)$, $(X, -K_X)$ degenerates to $(X_0, \mathcal{O}_{X_0}(2))$. Thus $\varepsilon(X, -K_X; 1) \geq \varepsilon(X_0, \mathcal{O}_{X_0}(2); 1) = \varepsilon(X_{2P}, L_{2P}; 1) \geq s_1(2P) = 2$.

Next, we think about the upper bounds. For No.5-10, it is known that X is covered by conics, i.e., for any general $p \in X$, there exists a smooth rational curve C containing p such that $C \cdot (-K_X) = 2$ (cf. [IP, Chapter 4]). Thus $\varepsilon(X, -K_X; 1) \leq 2$ in these cases. For No.11-15, $-K_X = 2H$ holds for the ample generator H . Assume that $\varepsilon(X, -K_X; 1) > 2$, i.e., $-K_{\tilde{X}} = \mu^*(-K_X) - 2E = 2(\mu^*H - E)$ is ample for the blowing up $\mu : \tilde{X} \rightarrow X$ at a

very general point $p \in X$ and $E = \mu^{-1}(p)$. Then \tilde{X} is a Fano 3-fold of index 2, i.e., a Del Pezzo 3-fold, and the Picard number is 2. By the classification of Del Pezzo manifolds (cf. [IP, §12.1]), $(-K_{\tilde{X}})^3 = 8(\pi^*H - E)^3$ must be $8 \cdot 6$ or $8 \cdot 7$, which contradicts $H^3 \leq 5$. Thus $\varepsilon(X, -K_X; 1) \leq 2$ holds for No.11-15. For No.16 and 17, X is covered by lines since X is a smooth quadric or \mathbb{P}^3 . Hence $\varepsilon(X, -K_X; 1) = r\varepsilon(X, H; 1) \leq r$ holds for the index r and the ample generator H for No.16 and 17.

Thus the inequalities in $(*)$ are in fact equalities, and the proof is completed. \square

5.3. Other examples. To apply Corollary 4.4, we have to find toric degenerations. We give some examples which degenerate to (unions of) toric varieties.

Example 5.8. Let G be a connected reductive group. Alexeev and Brion [AB] proved that any polarized spherical G -variety (X, L) admits a flat degeneration to a polarized toric variety over \mathbb{A}^1 and gave an explicit description of the moment polytope of the central fiber. Note that this degeneration is trivial over $\mathbb{A}^1 \setminus \{0\}$. Hence we can get a lower bound of $\varepsilon(X, L; 1)$ by applying Corollary 4.4 to this degeneration.

Example 5.9. For an n -dimensional polarized variety (X, L) and a flag Y_\bullet of subvarieties of X , that is, a chain $X = Y_0 \supset Y_1 \supset \cdots \supset Y_n$, where Y_i is a subvariety of codimension i in X which is nonsingular at the point Y_n , we can define the Okounkov body $\Delta_{Y_\bullet}(L) \subset \mathbb{R}^n$ (see [LM] or [KK]). Roughly, we define a graded semigroup $\Gamma \subset \mathbb{N} \times \mathbb{N}^n$ from (X, L) and Y_\bullet , and $\Delta(L) = \Delta_{Y_\bullet}(L)$ is defined to be the intersection of $\{1\} \times \mathbb{R}^n$ with the closure of the convex hull of Γ in $\mathbb{R} \times \mathbb{R}^n$. Note that $\Delta(L)$ is nothing but the moment polytope $\Delta(\Gamma)$ if Γ is finitely generated (cf. Definition 3.1). Anderson [An] showed that if Γ is finitely generated, (X, L) admits a flat degeneration to the not necessarily normal polarized toric variety $(X(\Gamma), L(\Gamma))$ over \mathbb{A}^1 which is trivial over $\mathbb{A}^1 \setminus \{0\}$. Thus $\varepsilon(X, L; 1) \geq \varepsilon(X_{\Delta(L)}, L_{\Delta(L)}; 1_{\Delta(L)})$ holds by Corollary 4.4 in this case. The author [It1] proved that $\varepsilon(X, L; 1) \geq \varepsilon(X_{\Delta(L)}, L_{\Delta(L)}; 1_{\Delta(L)})$ holds without the assumption that Γ is finitely generated if we define $\varepsilon(X_\Delta, L_\Delta; 1_\Delta)$ for any closed convex set $\Delta \subset \mathbb{R}^n$ suitably.

Example 5.10. (cf. [BBC+, 3.10]) Let P be an integral polytope of dimension n in $M_{\mathbb{R}}$. A polytope decomposition \mathcal{P} of P is a finite subset of $\{\sigma \mid \sigma \text{ is a polytope in } M_{\mathbb{R}}\}$ such that

- (i) $P = \bigcup_{\sigma \in \mathcal{P}} \sigma$,
- (ii) if $\sigma \in \mathcal{P}$ and τ is a face of σ , then $\tau \in \mathcal{P}$,
- (iii) if $\sigma, \sigma' \in \mathcal{P}$, then $\sigma \cap \sigma'$ is either a common face of σ, σ' or empty.

We say \mathcal{P} is integral (resp. rational) if all $\sigma \in \mathcal{P}$ are integral (resp. rational) polytopes. For example, a rational affine function $f : M_{\mathbb{R}} \rightarrow \mathbb{R}$ defines a rational polytope decomposition \mathcal{P}_f of P by $\mathcal{P}_f := \{\sigma \cap f^{-1}([0, +\infty)), \sigma \cap f^{-1}(0), \sigma \cap f^{-1}((-\infty, 0])\}_{\sigma \in \mathcal{P}}$.

Let \mathcal{P} be an integral polytope decomposition of P . If there exists a function $\varphi : P \rightarrow \mathbb{R}$ such that

- (a) φ is piecewise affine and strictly convex with respect to \mathcal{P} ,
- (b) φ takes integral values at all $u \in P \cap M$,

then one can construct an $n + 1$ dimensional toric variety \mathcal{X} , an ample line bundle \mathcal{L} on \mathcal{X} , and a projective toric morphism $f : \mathcal{X} \rightarrow \mathbb{A}^1$ such that $X_0 = \bigcup_{i=1}^r X_{P_i}$, $\mathcal{L}|_{X_{P_i}} = L_{P_i}$ and $X_t = X_P$, $L_t = L_P$ for any $t \in \mathbb{A}^1 \setminus \{0\}$, where $\{\sigma \in \mathcal{P} \mid \dim \sigma = n\} = \{P_1, \dots, P_r\}$. See [GS] for example. Thus in this case $L_P(s_1(P_1), \dots, s_1(P_r))$ is nef by Corollary 4.4. Such φ exists at least for the decomposition $k\mathcal{P}_f = \{k\sigma\}_{\sigma \in \mathcal{P}_f}$ of kP defined by a rational affine function f if $k \in \mathbb{N}$ is sufficiently large and divisible.

For example, Theorem 0.6 in [Ec], which states $\varepsilon(\mathbb{P}^2, \mathcal{O}(1); \overbrace{1, \dots, 1}^{10}) \geq 4/13$, follows from this argument by using his decomposition of $\text{conv}(0, e_1, e_2) \subset \mathbb{R}^2$ in his paper.

REFERENCES

- [AB] V. Alexeev and M. Brion, *Toric degenerations of spherical varieties*, Selecta Math. (N.S.) 10 (2004), no. 4, 453-478.
- [An] D. Anderson, *Okounkov bodies and toric degenerations*, arXiv:1001.4566
- [Bau] T. Bauer, *Seshadri constants and periods of polarized abelian varieties*, with an appendix by the author and Tomasz Szemberg. Math. Ann. 312 (1998), no. 4, 607-623.
- [BBC+] T. Bauer, C. Bocci, S. Cooper, S. Di Rocco, M. Dumnicki, B. Harbourne, K. Jabbusch, A.L. Knutsen, A. Küronya, R. Miranda, J. Roé, H. Schenck, T. Szemberg, and Z. Teitler, *Recent developments and open problems in linear series*, arXiv:1101.4363.
- [BDH+] T. Bauer, S. Di Rocco, B. Harbourne, M. Kapustka, A. Knutsen, W. Syzdek, and T. Szemberg, *A primer on Seshadri constants*, Interactions of classical and numerical algebraic geometry, 33-70, Contemp. Math., 496, Amer. Math. Soc., Providence, RI, 2009.
- [Bat] V.V. Batyrev, *Toric Fano threefolds*, Izv. Akad. Nauk SSSR Ser. Mat. 45 (1981), no. 4, 704-717, 927.
- [Bi] P. Biran, *Constructing new ample divisors out of old ones*, Duke Math. J. 98 (1999), no. 1, 113-135.
- [CI] J. Christophersen and N.O. Ilten, *Stanley-Reisner degenerations of Mukai varieties*, arXiv:1102.4521.
- [Deb] O. Debarre, *Higher-dimensional algebraic geometry*, Universitext. Springer-Verlag, New York, 2001. xiv+233 pp.
- [Dem] J.P. Demailly, *Singular Hermitian metrics on positive line bundles*, Complex algebraic varieties (Bayreuth, 1990), 87-104, Lecture Notes in Math., 1507, Springer, Berlin, 1992.
- [Di] S. Di rocco, *Generation of k -jets on toric varieties*, Math. Z. 231 (1999), no. 1, 169-188.
- [Ec] T. Eckl, *An asymptotic version of Dumnicki's algorithm for linear systems in \mathbb{CP}^2* , Geom. Dedicata 137 (2008), 149-162.
- [EKL] L. Ein, O. Küchle, and R. Lazarsfeld, *Local positivity of ample line bundles*, J. Differential Geom. 42 (1995), no. 2, 193-219.
- [Ei] D. Eisenbud, *Commutative Algebra, with a View Toward Algebraic Geometry*, Graduate Texts in Math, no.150, Springer-Verlag, New York, 1995.
- [Fu] W. Fulton, *Introduction to toric varieties*, Annals of Mathematics Studies, 131. Princeton University Press, Princeton, NJ, 1993. xii+157 pp.
- [Ga] S. Galkin, *Small toric degenerations of Fano threefolds*, <http://sergey.ipmu.jp/std.pdf>, 2008.
- [GS] M. Gross and B. Siebert, *An invitation to toric degenerations*, arXiv:0808.2749.
- [HW] J.-M. Hwang and J.H. Keum, *Seshadri-exceptional foliations*, Math. Ann. 325 (2003), no. 2, 287-297.
- [ILP] N.O. Ilten, J. Lewis, and V. Przyjalkowski, *Toric Degenerations of Fano Threefolds Giving Weak Landau-Ginzburg Models*, arXiv:1102.4664.
- [IP] V.A. Iskovskikh and Yu.G. Prokhorov, *Fano varieties. Algebraic geometry, V*, Encyclopaedia Math. Sci., 47, Springer, Berlin, 1999.
- [It1] A. Ito, *Okounkov bodies and Seshadri constants*, preprint.
- [It2] A. Ito, *Algebraic-geometric characterization of Cayley polytopes*, preprint.
- [KK] K. Kaveh and A. G. Khovanskii, *Newton convex bodies, semigroups of integral points, graded algebras and intersection theory*, arXiv:0904.3350.
- [La1] R. Lazarsfeld, *Lengths of periods and Seshadri constants of abelian varieties*, Math. Res. Lett. 3 (1996), no. 4, 439-447.
- [La2] R. Lazarsfeld, *Positivity in algebraic geometry I*, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 48. Springer, Berlin (2004).
- [LM] R. Lazarsfeld and M. Mustață, *Convex bodies associated to linear series*, Ann. Sci. Ec. Norm. Super. (4) 42 (2009), no. 5, 783-835.
- [MP] D. McDuff and L. Polterovich, *Symplectic packings and algebraic geometry*, Invent. Math. 115 (1994), no. 3, 405-434.
- [Na1] M. Nakamaye, *Seshadri constants on abelian varieties*, Amer. J. Math. 118 (1996), no. 3, 621-635.

- [Na2] M. Nakamaye, *Seshadri constants and the geometry of surfaces*, J. Reine Angew. Math. 564 (2003), 205-214.
- [RT] J. Ross and R. Thomas, *A study of the Hilbert-Mumford criterion for the stability of projective varieties*, J. Algebraic Geom. 16 (2007), no. 2, 201-255.
- [St] A. Steffens, *Remarks on Seshadri constants*, Math. Z. 227 (1998), no. 3, 505-510.
- [WW] K. Watanabe and M. Watanabe, *The classification of Fano 3-folds with torus embeddings*, Tokyo J. Math. 5 (1982), no. 1, 37-48.

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